OSNOVI METODE KONAČNIH ELEMENATA Predavanje VII

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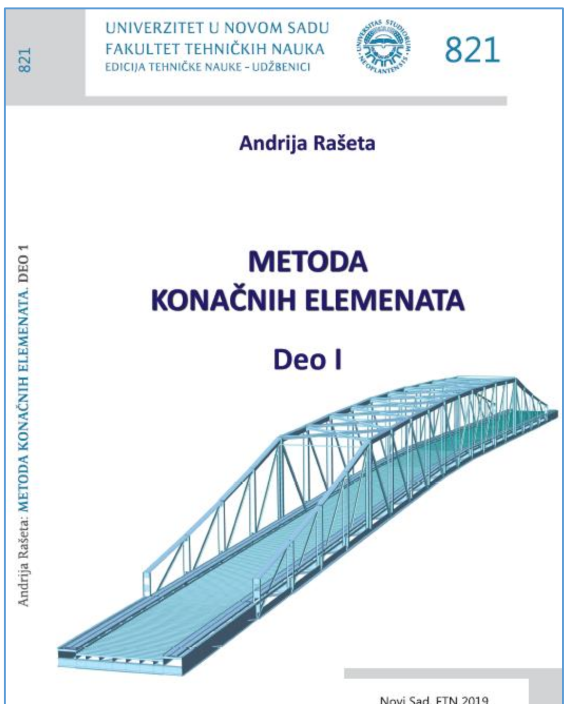
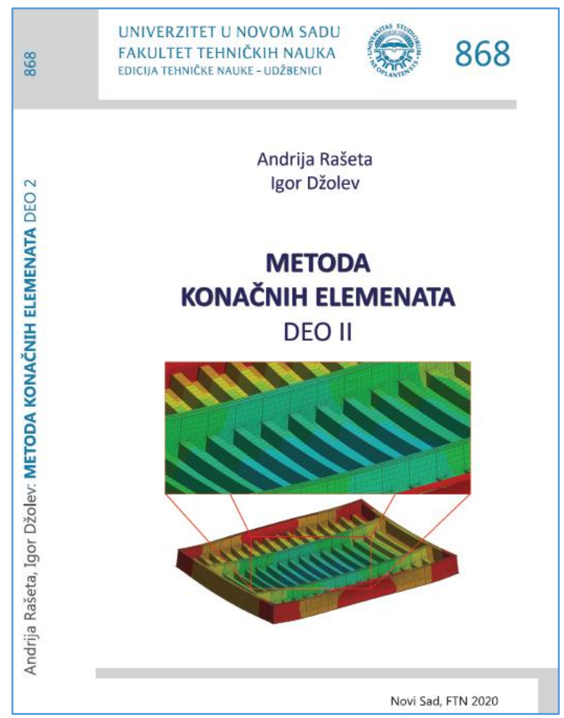
3D problemi
Prirodne koordinate i interpolacione funkcije
KE oblika tetraedra
KE oblika heksaedra
Izoparametarski 3D KE
Linearna statička analiza

Literatura

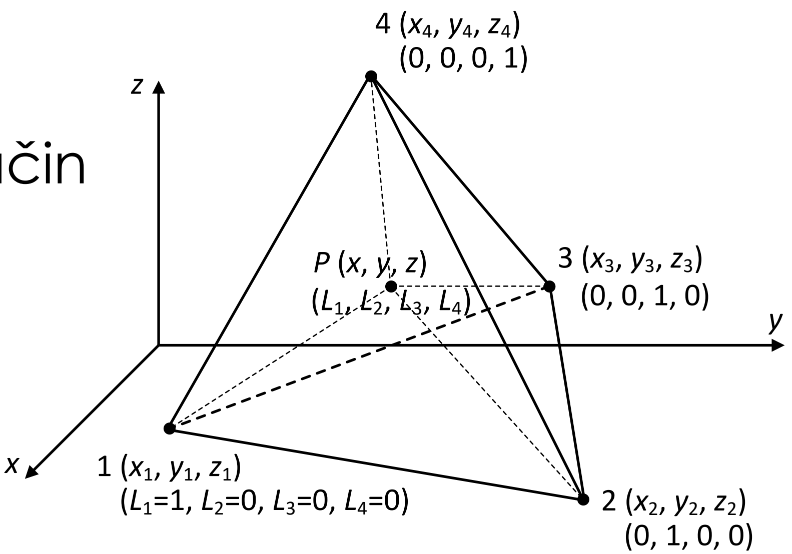
- **Metoda konačnih elemenata, deo I**, A. Rašeta, FTN Novi Sad, 2019.
- **Metoda konačnih elemenata, deo II**, A. Rašeta, I. Džolev, FTN Novi Sad, 2020.





Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

- **KE oblika tetraedra**
- Čvorovi tetraedra u Dekartovom koordinatnom sistemu desne orijentacije obeleženi su tako da rastu suprotno od smera obrtanja kazaljke na časovniku gledajući sa strane čvora 4
- Položaj proizvoljne tačke u zavisnosti od koordinata temena može da se prikaže na sledeći način



$$x = L_1x_1 + L_2x_2 + L_3x_3 + L_4x_4$$
$$y = L_1y_1 + L_2y_2 + L_3y_3 + L_4y_4$$
$$z = L_1z_1 + L_2z_2 + L_3z_3 + L_4z_4$$

  - pri čemu **zapreminske koordinate** nisu međusobno nezavisne

$$L_1 + L_2 + L_3 + L_4 = 1$$

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

- **KE oblika tetraedra**
- Prethodni izrazi u matričnom obliku glase

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}$$

- odakle sledi

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} = \frac{1}{6V} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} \qquad V = \frac{1}{6} \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} = \frac{1}{6}(a_1 + a_2 + a_3 + a_4)$$

$$L_i = \frac{1}{6V}(a_i + b_i x + c_i y + d_i z), \qquad i = 1,2,3,4$$

$$a_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}, \; b_1 = -\begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}, \; c_1 = \begin{vmatrix} 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix}, \; d_1 = -\begin{vmatrix} 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix}$$

$$a_2 = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}, \; b_2 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}, \; c_2 = -\begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix}, \; d_2 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix}$$

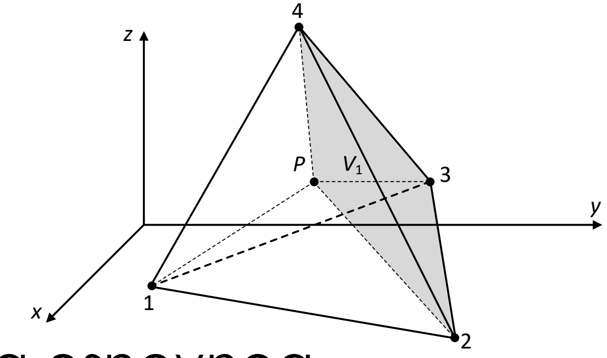
$$a_3 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix}, \; b_3 = -\begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_4 & z_4 \end{vmatrix}, \; c_3 = \begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_4 & z_4 \end{vmatrix}, \; d_3 = -\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_4 & y_4 \end{vmatrix}$$

$$a_4 = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}, \; b_4 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix}, \; c_4 = -\begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{vmatrix}, \; d_4 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

- **KE oblika tetraedra**
- Izraz u zagradi uz koeficijent 1/6 za i = 1 predstavlja zapreminu dela tetraedra definisanog koordinatama čvorova 2(x_2, y_2), 3(x_3, y_3), 4(x_4, y_4) i tačke P(x, y, z) pa se na osnovu toga zaključuje da zapreminska koordinata L_1 predstavlja odnos zapremine dela tetraedra definisanog čvorovima P, 2, 3, 4 naspram čvora 1 i zapremine V celog tetraedra

$$L_1 = \frac{V_1}{V} \qquad L_2 = \frac{V_2}{V} \qquad L_3 = \frac{V_3}{V} \qquad L_4 = \frac{V_4}{V}$$



- S obzirom na to da se vrednost zapreminske koordinate Li linearno menja od 1 u i-tom čvoru (temenu) do 0 u preostalim čvorovima (temenima) ona je jednaka IF i-tog čvora osnovnog tetraedarskog KE (linearna interpolacija), odnosno važi sledeće

$$N_1 = L_1 \qquad N_2 = L_2 \qquad N_3 = L_3 \qquad N_4 = L_4$$

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

- **KE oblika tetraedra**
- Ako je *f* funkcija od L_1, L_2, L_3 i L_4 operacija parcijalnog diferenciranja po koordinatama x, *y* i z obavlja se na sledeći način

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{4} \frac{\partial f}{\partial L_i}\frac{\partial L_i}{\partial x}, \qquad \frac{\partial f}{\partial y} = \sum_{i=1}^{4} \frac{\partial f}{\partial L_i}\frac{\partial L_i}{\partial y}, \qquad \frac{\partial f}{\partial z} = \sum_{i=1}^{4} \frac{\partial f}{\partial L_i}\frac{\partial L_i}{\partial z}$$

$$\frac{\partial L_i}{\partial x} = \frac{b_i}{6V}, \qquad \frac{\partial L_i}{\partial y} = \frac{c_i}{6V}, \qquad \frac{\partial L_i}{\partial z} = \frac{d_i}{6V}, \qquad i = 1,2,3,4$$

- Integracija po zapremini KE kada se pod integralom javljaju prirodne koordinate obavlja se pomoću izraza

$$\int_V L_1^i(x,y,z) L_2^j(x,y,z) L_3^k(x,y,z) L_4^l(x,y,z) dV = \frac{i!\,j!\,k!\,l!}{(i+j+k+l+3)!} 6V$$

  - gde su *i*, *j*, *k* i *l* celobrojni eksponenti

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

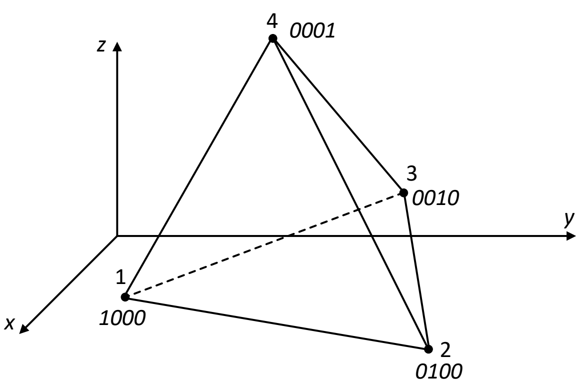
- **KE oblika tetraedra**
- **IF mogu da se odrede pomoću formule**

$$N_{\alpha\beta\gamma\delta}(L_1, L_2, L_3, L_4) = N_\alpha(L_1)N_\beta(L_2)N_\gamma(L_3)N_\delta(L_4)$$

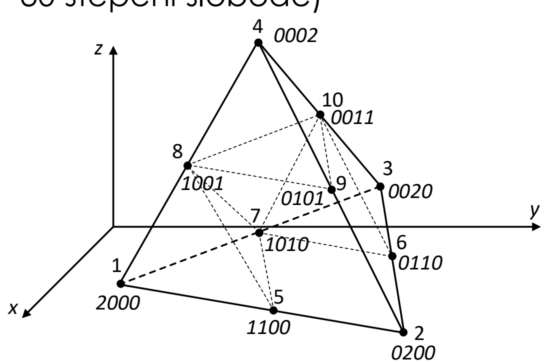
$$N_\alpha(L_1) = \begin{cases} \prod_{i=1}^{\alpha} \left(\dfrac{nL_1 - i + 1}{i}\right) & \text{za} \ \ \alpha \geq 1 \\ 1 & \text{za} \ \ \alpha = 0 \end{cases}$$

  - pri čemu je *n* stepen polinoma
  - Analogni izrazi važe za $N_\beta(L_2)$, $N_\gamma(L_3)$ i $N_\delta(L_4)$
  - Čvorovima KE pridružuju se četiri broja čiji je zbir jednak stepenu interpolacionog polinoma

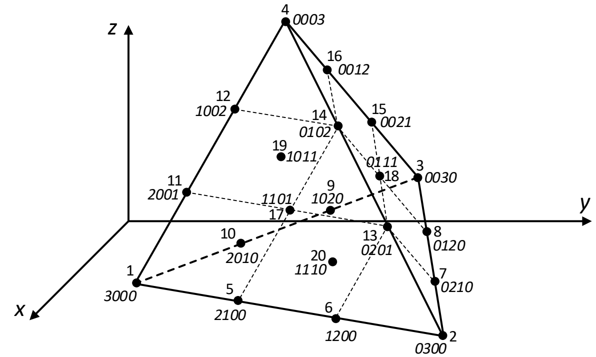
Linearna interpolacija (osnovni element; 4 čvora; 12 stepeni slobode)



Kvadratna interpolacija (element drugog reda; 10 čvorova (u temenima i sredinama izvodnica), 30 stepeni slobode)



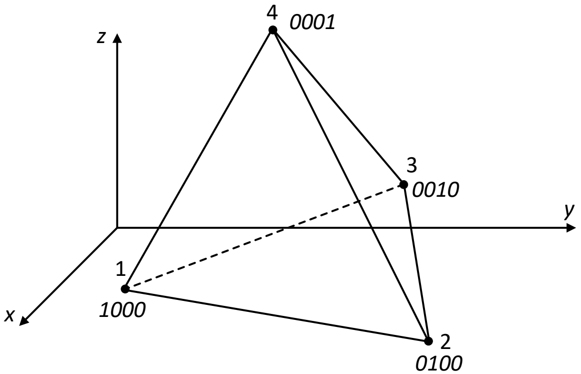
Kubna interpolacija (element trećeg reda; 20 čvorova (u temenima, trećinama izvodnica i težištima površina strana), 60 stepeni slobode)



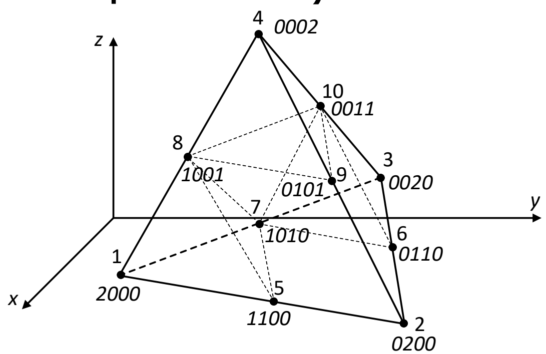
Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

- **KE oblika tetraedra**
- **IF mogu da se odrede pomoću formule**
  - Svaki broj se odnosi na jednu od stranica tetraedra. Prvi broj je jednak 0 u čvorovima koji leže na površini strane gde je $L_1 = 0$ (naspram čvora 1), drugi broj je jednak 0 u čvorovima koji leže na površini strane gde je $L_2 = 0$ (naspram čvora 2), treći broj je jednak 0 u čvorovima koji leže na površini strane gde je $L_3 = 0$ (naspram čvora 3) i četvrti broj je jednak 0 u čvorovima koji leže na površini strane gde je $L_4 = 0$ (naspram čvora 4). Svaki od brojeva koji leži na paralelnim ravnima (koje su na jednakim međusobnim rastojanjima) u odnosu na odgovarajuću stranu tetraedra rastu sa udaljavanjem od ravni na kojoj je odgovarajuća zapreminska koordinata jednaka nuli

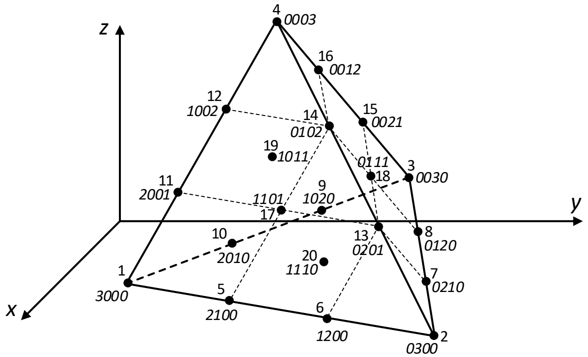
Linearna interpolacija (osnovni element; 4 čvora; 12 stepeni slobode)



Kvadratna interpolacija (element drugog reda; 10 čvorova (u temenima i sredinama izvodnica), 30 stepeni slobode)



Kubna interpolacija (element trećeg reda; 20 čvorova (u temenima, trećinama izvodnica i težištima površina strana), 60 stepeni slobode)

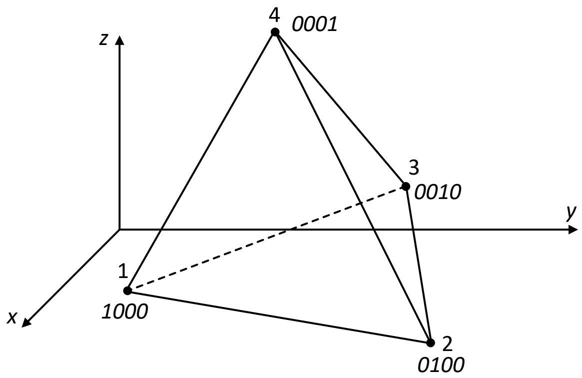


Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

- **KE oblika tetraedra**
- **Linearne IF**
  - čvor *1000*

$$N_\alpha(L_1) = N_{\alpha=1}(L_1) = \left(\frac{1 \cdot L_1 - 1 + 1}{1}\right)$$
$$N_\beta(L_2) = N_{\beta=0}(L_2) = 1$$
$$N_\gamma(L_3) = N_{\gamma=0}(L_3) = 1$$
$$N_\delta(L_4) = N_{\delta=0}(L_4) = 1$$

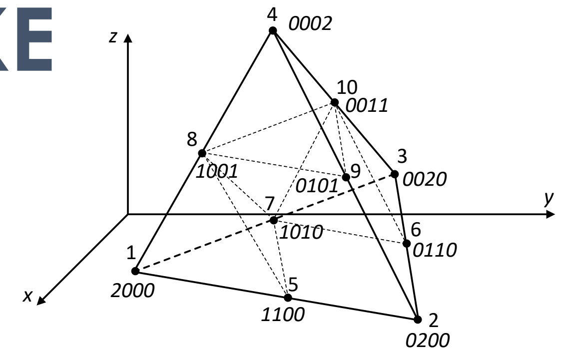
$$N_1 = L_1$$



  - Analognim postupkom određuju se ostale IF

$$N_2 = L_2 \qquad N_3 = L_3 \qquad N_4 = L_4$$

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE



- **KE oblika tetraedra**
- **Kvadratne IF**
  - čvor *2000*

$$N_\alpha(L_1) = N_{\alpha=2}(L_1) = \left(\frac{2\cdot L_1 - 1 + 1}{1}\right)\left(\frac{2\cdot L_1 - 2 + 1}{2}\right)$$
$$N_\beta(L_2) = N_{\beta=0}(L_2) = 1$$
$$N_\gamma(L_3) = N_{\gamma=0}(L_3) = 1$$
$$N_\delta(L_4) = N_{\delta=0}(L_4) = 1$$

$$N_1 = L_1(2L_1 - 1)$$

  - čvor *1100*

$$N_\alpha(L_1) = N_{\alpha=1}(L_1) = \left(\frac{2\cdot L_1 - 1 + 1}{1}\right)$$
$$N_\beta(L_2) = N_{\beta=1}(L_2) = \left(\frac{2\cdot L_2 - 1 + 1}{1}\right)$$
$$N_\gamma(L_3) = N_{\gamma=0}(L_3) = 1$$
$$N_\delta(L_4) = N_{\delta=0}(L_4) = 1$$

$$N_5 = 4L_1 4L_2$$

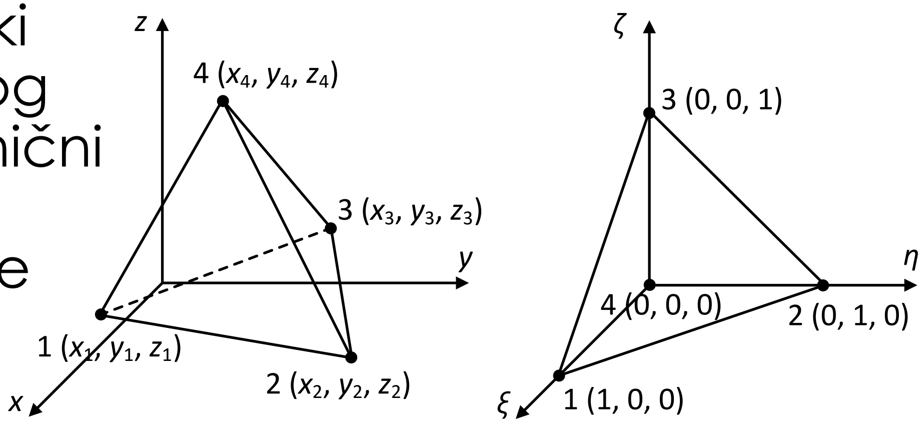
  - Analognim postupkom određuju se ostale IF

$$N_2 = L_2(2L_2 - 1) \qquad N_3 = L_3(2L_3 - 1) \qquad N_4 = L_4(2L_4 - 1) \qquad N_6 = 4L_2 4L_3$$

$$N_7 = 4L_2 4L_3 \qquad N_8 = 4L_1 4L_3 \qquad N_9 = 4L_2 4L_4 \qquad N_{10} = 4L_3 4L_4$$

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

- **KE oblika tetraedra**
- Analogno kao i kod dvodimenzionalnih elemenata pored prethodno opisanih prirodnih zapreminskih koordinata može da se koristi **sistem prirodnih koordinata *ξ*, *η* i *ζ***
- Proizvoljni osnovni tetraedarski KE iz Dekartovog koordinatnog sistema preslikava se na jedinični tetraedar u prirodnom koordinatnom sistemu, tj. važe sledeće veze za linearnu transformaciju



$$x = x_4 + (x_1 - x_4)\xi + (x_2 - x_4)\eta + (x_3 - x_4)\zeta$$
$$y = y_4 + (y_1 - y_4)\xi + (y_2 - y_4)\eta + (y_3 - y_4)\zeta$$
$$z = z_4 + (z_1 - z_4)\xi + (z_2 - z_4)\eta + (z_3 - z_4)\zeta$$

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

- **KE oblika tetraedra**
- Ako je *f* funkcija od x, *y* i z koje su u funkciji od ξ, η i ζ, tj. $f = f(x(\xi,\eta,\zeta), y(\xi,\eta,\zeta), z(\xi,\eta,\zeta))$ operacija parcijalnog diferenciranja obavlja se na sledeći način

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \xi}$$
$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \eta}$$
$$\frac{\partial f}{\partial \zeta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \zeta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \zeta}$$

$$\begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \\ \frac{\partial f}{\partial \zeta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{Bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} = \begin{bmatrix} x_1 - x_4 & x_2 - x_4 & x_3 - x_4 \\ y_1 - y_4 & y_2 - y_4 & y_3 - y_4 \\ z_1 - z_4 & z_2 - z_4 & z_3 - z_4 \end{bmatrix}$$

$$\begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \\ \frac{\partial f}{\partial \zeta} \end{Bmatrix}$$

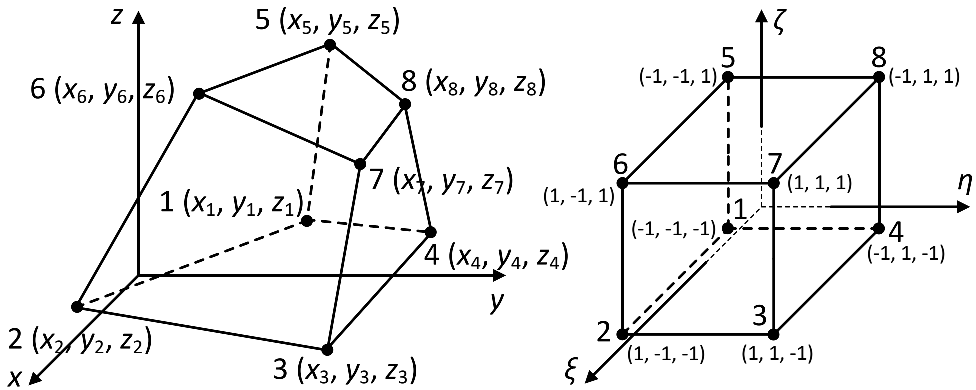
$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22}J_{33} - J_{23}J_{32} & J_{13}J_{32} - J_{12}J_{33} & J_{12}J_{23} - J_{13}J_{22} \\ J_{23}J_{31} - J_{21}J_{33} & J_{11}J_{33} - J_{13}J_{31} & J_{13}J_{21} - J_{11}J_{23} \\ J_{21}J_{32} - J_{22}J_{31} & J_{12}J_{31} - J_{11}J_{32} & J_{11}J_{22} - J_{12}J_{21} \end{bmatrix}$$

$$\det \mathbf{J} = -J_{13}J_{22}J_{31} + J_{12}J_{23}J_{31} + J_{13}J_{21}J_{32} - J_{11}J_{23}J_{32} - J_{12}J_{21}J_{33} + J_{11}J_{22}J_{33}$$

$$V = \frac{1}{6}\det \mathbf{J} \qquad L_1 = \xi, \quad L_2 = \eta, \quad L_3 = \zeta, \quad L_4 = 1 - \xi - \eta - \zeta \qquad dxdydz = \det \mathbf{J}\, d\xi d\eta d\zeta$$

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

- **KE oblika heksaedra**
- Uvodi se sistem **prirodnih koordinata *ξ*, *η* i *ζ*** koji je analogan sistemu prirodnih koordinata za element u obliku četvorougla



- Dekartove koordinate x, y i z proizvoljne tačke u polju KE mogu da se prikažu kao linearna kombinacija koordinata čvorova x_1, y_1, ..., x_8 i y_8

$$x = \sum_{i=1}^{8} L_i x_i \qquad y = \sum_{i=1}^{8} L_i y_i \qquad z = \sum_{i=1}^{8} L_i z_i$$

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

- **KE oblika heksaedra**
- Za određivanje funkcija L_i koristi se veza između globalnih Dekartovih i prirodnih (lokalnih bezdimenzionalnih) koordinata

$$x = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\zeta + \alpha_5\xi\eta + \alpha_6\xi\zeta + \alpha_7\eta\zeta + \alpha_8\xi\eta\zeta$$

$$x = \mathbf{A}\boldsymbol{\alpha} \qquad \mathbf{A} = [1 \quad \xi \quad \eta \quad \zeta \quad \xi\eta \quad \xi\zeta \quad \eta\zeta \quad \xi\eta\zeta]$$

$$\boldsymbol{\alpha}^T = \{\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7 \quad \alpha_8\}$$

- Za temena heksaedra važi

$$x = x_i,\ \ \xi = \xi_i,\ \ \eta = \eta_i, \qquad i = 1,2,3,\ldots,8$$

- Koristeći prethodne za svaki čvor KE sledi

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix}$$

$$\mathbf{X} = \mathbf{E}\boldsymbol{\alpha} \qquad \boldsymbol{\alpha} = \mathbf{E}^{-1}\mathbf{X} \qquad x = \mathbf{A}\mathbf{E}^{-1}\mathbf{X} \qquad \mathbf{L} = \mathbf{A}\mathbf{E}^{-1}$$

$$\mathbf{E}^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

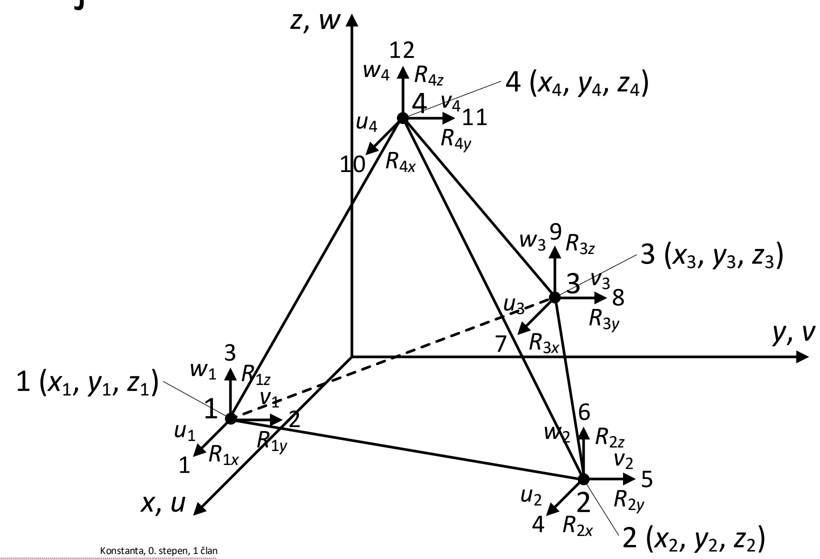
$$L_i = \frac{1}{8}(1 + \xi_i\xi)(1 + \eta_i\eta)(1 + \zeta_i\zeta), \qquad i = 1,2,3,\ldots,8$$

KE oblika tetraedra. Linearna interpolacija. CSTh KE

- Analizira se KE oblika tetraedra sa čvorovima u temenima koji su obeleženi tako da njihovi brojevi 1, 2 i 3 rastu u smeru suprotnom od smera obrtanja kazaljke na časovniku posmatrajući sa strane čvora 4

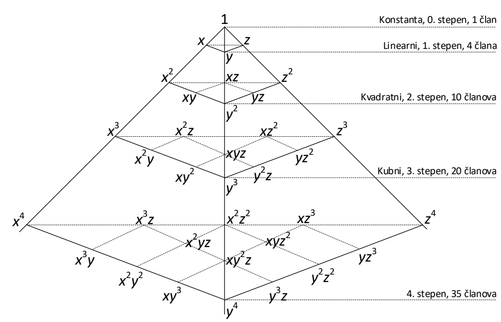
$$\mathbf{d}^T = \{\mathbf{d}_1 \quad \mathbf{d}_2 \quad \mathbf{d}_3 \quad \mathbf{d}_4\}$$

$$\mathbf{d}_i^T = \{u_i \quad v_i \quad w_i\}, \qquad i = 1,2,3,4$$



- Raspodela pomeranja u polju KE definisana je potpunim polinomima prvog stepena

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z$$
$$v = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 z$$
$$w = \alpha_9 + \alpha_{10} x + \alpha_{11} y + \alpha_{12} z$$



KE oblika tetraedra. Linearna interpolacija. CSTh KE

- IF. Direktan postupak

$$\boldsymbol{\alpha}^T = \{\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7 \quad \alpha_8 \quad \alpha_9 \quad \alpha_{10} \quad \alpha_{11} \quad \alpha_{12}\}$$

$$\mathbf{u} = \mathbf{A}\boldsymbol{\alpha} \to \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} 1 & x & y & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & z \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{12} \end{Bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 1 & x & y & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & z \end{bmatrix}$$

  - Za čvorove KE (granični uslovi su: $u = u_i$, $v = v_i$, $w = w_i$ i koordinate čvorova su: $x = x_i$, $y = y_i$, $z = z_i$, gde je i = 1, 2,3, i 4) sledi

$$\mathbf{d} = \mathbf{C}\boldsymbol{\alpha} \to \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & z_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_2 & y_2 & z_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_3 & y_3 & z_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_4 & y_4 & z_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_4 & y_4 & z_4 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \\ \alpha_{11} \\ \alpha_{12} \end{Bmatrix}$$

itd.

KE oblika tetraedra. Linearna interpolacija. CSTh KE

- IF. Zapreminske koordinate

$$\mathbf{N} = \mathbf{AC}^{-1} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix}$$

$$N_i = \frac{1}{6V}\left(a_i + b_i x + c_i y + d_i z\right), \qquad i = 1,2,3,4$$

$$V = \frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} = \frac{1}{6}\left(a_1 + a_2 + a_3 + a_4\right)$$

$$a_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix},\ b_1 = -\begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix},\ c_1 = \begin{vmatrix} 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix},\ d_1 = -\begin{vmatrix} 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix}$$

$$a_2 = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix},\ b_2 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix},\ c_2 = -\begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix},\ d_2 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix}$$

$$a_3 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix},\ b_3 = -\begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_4 & z_4 \end{vmatrix},\ c_3 = \begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_4 & z_4 \end{vmatrix},\ d_3 = -\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_4 & y_4 \end{vmatrix}$$

$$a_4 = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix},\ b_4 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix},\ c_4 = -\begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{vmatrix},\ d_4 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

KE oblika tetraedra. Linearna interpolacija. CSTh KE

- Matrica **B**

$$\mathbf{B} = \mathbf{D}_k \mathbf{N}$$

$$\mathbf{D}_k = \begin{bmatrix} \partial/\partial_x & 0 & 0 \\ 0 & \partial/\partial_y & 0 \\ 0 & 0 & \partial/\partial_z \\ \partial/\partial_y & \partial/\partial_x & 0 \\ 0 & \partial/\partial_z & \partial/\partial_y \\ \partial/\partial_z & 0 & \partial/\partial_x \end{bmatrix}$$

$$\mathbf{B} = \frac{1}{6V}\begin{bmatrix} b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 & 0 & b_4 & 0 & 0 \\ 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 & c_4 & 0 \\ 0 & 0 & d_1 & 0 & 0 & d_2 & 0 & 0 & d_3 & 0 & 0 & d_4 \\ c_1 & b_1 & 0 & c_2 & b_2 & 0 & c_3 & b_3 & 0 & c_4 & b_4 & 0 \\ 0 & d_1 & c_1 & 0 & d_2 & c_2 & 0 & d_3 & c_3 & 0 & d_4 & c_4 \\ d_1 & 0 & b_1 & d_2 & 0 & b_2 & d_3 & 0 & b_3 & d_4 & 0 & b_4 \end{bmatrix}$$

- Matrica krutosti

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} dV = \mathbf{B}^T \mathbf{D} \mathbf{B} V$$

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix}$$

Komentari:

- Linearna raspodela komponenata pomeranja po spoljašnjim površinama tetraedra može jednoznačno da se odredi sa tri stepena slobode pomeranja u čvorovima pa se zaključuje da je kompatibilnost pomeranja na površinama između susednih KE zadovoljena, tj. element **spada u konformne KE**
- Takođe, KE **ispunjava i ostale uslove za monotonu konvergenciju** rešenja (interpolacione funkcije mogu da opišu pomeranje kao krutog tela i polje konstantnih deformacija)
- S obzirom na to da su elementi matrice B konstantni sledi da su komponente deformacije u KE konstantne pa se element naziva **tetraedar sa konstantnim deformacijama ili CSTh element** (Constant Strain Tetrahedral element)

KE oblika tetraedra. Linearna interpolacija. CSTh KE

- Matrica krutosti

$$\mathbf{k} = \frac{1}{36V} \cdot$$

kolone 1 do 3

$b_1^2D_{11}+c_1^2D_{44}+d_1^2D_{66}$	$b_1c_1\left(D_{12}+D_{44}\right)$	$b_1d_1\left(D_{13}+D_{66}\right)$
$b_1c_1\left(D_{21}+D_{44}\right)$	$c_1^2D_{22}+b_1^2D_{44}+d_1^2D_{55}$	$c_1d_1\left(D_{23}+D_{55}\right)$
$b_1d_1\left(D_{31}+D_{66}\right)$	$c_1d_1\left(D_{32}+D_{55}\right)$	$d_1^2D_{33}+c_1^2D_{55}+b_1^2D_{66}$
$b_1b_2D_{11}+c_1c_2D_{44}+d_1d_2D_{66}$	$b_2c_1D_{12}+b_1c_2D_{44}$	$b_2d_1D_{13}+b_1d_2D_{66}$
$b_1c_2D_{21}+b_2c_1D_{44}$	$c_1c_2D_{22}+b_1b_2D_{44}+d_1d_2D_{55}$	$c_2d_1D_{23}+c_1d_2D_{55}$
$b_1d_2D_{31}+b_2d_1D_{66}$	$c_1d_2D_{32}+c_2d_1D_{55}$	$d_1d_2D_{33}+c_1c_2D_{55}+b_1b_2D_{66}$
$b_1b_3D_{11}+c_1c_3D_{44}+d_1d_3D_{66}$	$b_3c_1D_{12}+b_1c_3D_{44}$	$b_3d_1D_{13}+b_1d_3D_{66}$
$b_1c_3D_{21}+b_3c_1D_{44}$	$c_1c_3D_{22}+b_1b_3D_{44}+d_1d_3D_{55}$	$c_3d_1D_{23}+c_1d_3D_{55}$
$b_1d_3D_{31}+b_3d_1D_{66}$	$c_1d_3D_{32}+c_3d_1D_{55}$	$d_1d_3D_{33}+c_1c_3D_{55}+b_1b_3D_{66}$
$b_1b_4D_{11}+c_1c_4D_{44}+d_1d_4D_{66}$	$b_4c_1D_{12}+b_1c_4D_{44}$	$b_4d_1D_{13}+b_1d_4D_{66}$
$b_1c_4D_{21}+b_4c_1D_{44}$	$c_1c_4D_{22}+b_1b_4D_{44}+d_1d_4D_{55}$	$c_4d_1D_{23}+c_1d_4D_{55}$
$b_1d_4D_{31}+b_4d_1D_{66}$	$c_1d_4D_{32}+c_4d_1D_{55}$	$d_1d_4D_{33}+c_1c_4D_{55}+b_1b_4D_{66}$

kolone 4 do 6

$b_1b_2D_{11}+c_1c_2D_{44}+d_1d_2D_{66}$	$b_1c_2D_{12}+b_2c_1D_{44}$	$b_1d_2D_{13}+b_2d_1D_{66}$
$b_2c_1D_{21}+b_1c_2D_{44}$	$c_1c_2D_{22}+b_1b_2D_{44}+d_1d_2D_{55}$	$c_1d_2D_{23}+c_2d_1D_{55}$
$b_2d_1D_{31}+b_1d_2D_{66}$	$c_2d_1D_{32}+c_1d_2D_{55}$	$d_1d_2D_{33}+c_1c_2D_{55}+b_1b_2D_{66}$
$b_2^2D_{11}+c_2^2D_{44}+d_2^2D_{66}$	$b_2c_2\left(D_{12}+D_{44}\right)$	$b_2d_2\left(D_{13}+D_{66}\right)$
$b_2c_2\left(D_{21}+D_{44}\right)$	$c_2^2D_{22}+b_2^2D_{44}+d_2^2D_{55}$	$c_2d_2\left(D_{23}+D_{55}\right)$
$b_2d_2\left(D_{31}+D_{66}\right)$	$c_2d_2\left(D_{32}+D_{55}\right)$	$d_2^2D_{33}+c_2^2D_{55}+b_2^2D_{66}$
$b_2b_3D_{11}+c_2c_3D_{44}+d_2d_3D_{66}$	$b_3c_2D_{12}+b_2c_3D_{44}$	$b_3d_2D_{13}+b_2d_3D_{66}$
$b_2c_3D_{21}+b_3c_2D_{44}$	$c_2c_3D_{22}+b_2b_3D_{44}+d_2d_3D_{55}$	$c_3d_2D_{23}+c_2d_3D_{55}$
$b_2d_3D_{31}+b_3d_2D_{66}$	$c_2d_3D_{32}+c_3d_2D_{55}$	$d_2d_3D_{33}+c_2c_3D_{55}+b_2b_3D_{66}$
$b_2b_4D_{11}+c_2c_4D_{44}+d_2d_4D_{66}$	$b_4c_2D_{12}+b_2c_4D_{44}$	$b_4d_2D_{13}+b_2d_4D_{66}$
$b_2c_4D_{21}+b_4c_2D_{44}$	$c_2c_4D_{22}+b_2b_4D_{44}+d_2d_4D_{55}$	$c_4d_2D_{23}+c_2d_4D_{55}$
$b_2d_4D_{31}+b_4d_2D_{66}$	$c_2d_4D_{32}+c_4d_2D_{55}$	$d_2d_4D_{33}+c_2c_4D_{55}+b_2b_4D_{66}$

kolone 7 do 9

$b_1b_3D_{11}+c_1c_3D_{44}+d_1d_3D_{66}$	$b_1c_3D_{12}+b_3c_1D_{44}$	$b_1d_3D_{13}+b_3d_1D_{66}$
$b_3c_1D_{21}+b_1c_3D_{44}$	$c_1c_3D_{22}+b_1b_3D_{44}+d_1d_3D_{55}$	$c_1d_3D_{23}+c_3d_1D_{55}$
$b_3d_1D_{31}+b_1d_3D_{66}$	$c_3d_1D_{32}+c_1d_3D_{55}$	$d_1d_3D_{33}+c_1c_3D_{55}+b_1b_3D_{66}$
$b_2b_3D_{11}+c_2c_3D_{44}+d_2d_3D_{66}$	$b_2c_3D_{12}+b_3c_2D_{44}$	$b_2d_3D_{13}+b_3d_2D_{66}$
$b_3c_2D_{21}+b_2c_3D_{44}$	$c_2c_3D_{22}+b_2b_3D_{44}+d_2d_3D_{55}$	$c_2d_3D_{23}+c_3d_2D_{55}$
$b_3d_2D_{31}+b_2d_3D_{66}$	$c_3d_2D_{32}+c_2d_3D_{55}$	$d_2d_3D_{33}+c_2c_3D_{55}+b_2b_3D_{66}$
$b_3^2D_{11}+c_3^2D_{44}+d_3^2D_{66}$	$b_3c_3\left(D_{12}+D_{44}\right)$	$b_3d_3\left(D_{13}+D_{66}\right)$
$b_3c_3\left(D_{21}+D_{44}\right)$	$c_3^2D_{22}+b_3^2D_{44}+d_3^2D_{55}$	$c_3d_3\left(D_{23}+D_{55}\right)$
$b_3d_3\left(D_{31}+D_{66}\right)$	$c_3d_3\left(D_{32}+D_{55}\right)$	$d_3^2D_{33}+c_3^2D_{55}+b_3^2D_{66}$
$b_3b_4D_{11}+c_3c_4D_{44}+d_3d_4D_{66}$	$b_4c_3D_{12}+b_3c_4D_{44}$	$b_4d_3D_{13}+b_3d_4D_{66}$
$b_3c_4D_{21}+b_4c_3D_{44}$	$c_3c_4D_{22}+b_3b_4D_{44}+d_3d_4D_{55}$	$c_4d_3D_{23}+c_3d_4D_{55}$
$b_3d_4D_{31}+b_4d_3D_{66}$	$c_3d_4D_{32}+c_4d_3D_{55}$	$d_3d_4D_{33}+c_3c_4D_{55}+b_3b_4D_{66}$

kolone 10 do 12

$b_1b_4D_{11}+c_1c_4D_{44}+d_1d_4D_{66}$	$b_1c_4D_{12}+b_4c_1D_{44}$	$b_1d_4D_{13}+b_4d_1D_{66}$
$b_4c_1D_{21}+b_1c_4D_{44}$	$c_1c_4D_{22}+b_1b_4D_{44}+d_1d_4D_{55}$	$c_1d_4D_{23}+c_4d_1D_{55}$
$b_4d_1D_{31}+b_1d_4D_{66}$	$c_4d_1D_{32}+c_1d_4D_{55}$	$d_1d_4D_{33}+c_1c_4D_{55}+b_1b_4D_{66}$
$b_2b_4D_{11}+c_2c_4D_{44}+d_2d_4D_{66}$	$b_2c_4D_{12}+b_4c_2D_{44}$	$b_2d_4D_{13}+b_4d_2D_{66}$
$b_4c_2D_{21}+b_2c_4D_{44}$	$c_2c_4D_{22}+b_2b_4D_{44}+d_2d_4D_{55}$	$c_2d_4D_{23}+c_4d_2D_{55}$
$b_4d_2D_{31}+b_2d_4D_{66}$	$c_4d_2D_{32}+c_2d_4D_{55}$	$d_2d_4D_{33}+c_2c_4D_{55}+b_2b_4D_{66}$
$b_3b_4D_{11}+c_3c_4D_{44}+d_3d_4D_{66}$	$b_3c_4D_{12}+b_4c_3D_{44}$	$b_3d_4D_{13}+b_4d_3D_{66}$
$b_4c_3D_{21}+b_3c_4D_{44}$	$c_3c_4D_{22}+b_3b_4D_{44}+d_3d_4D_{55}$	$c_3d_4D_{23}+c_4d_3D_{55}$
$b_4d_3D_{31}+b_3d_4D_{66}$	$c_4d_3D_{32}+c_3d_4D_{55}$	$d_3d_4D_{33}+c_3c_4D_{55}+b_3b_4D_{66}$
$b_4^2D_{11}+c_4^2D_{44}+d_4^2D_{66}$	$b_4c_4\left(D_{12}+D_{44}\right)$	$b_4d_4\left(D_{13}+D_{66}\right)$
$b_4c_4\left(D_{21}+D_{44}\right)$	$c_4^2D_{22}+b_4^2D_{44}+d_4^2D_{55}$	$c_4d_4\left(D_{23}+D_{55}\right)$
$b_4d_4\left(D_{31}+D_{66}\right)$	$c_4d_4\left(D_{32}+D_{55}\right)$	$d_4^2D_{33}+c_4^2D_{55}+b_4^2D_{66}$

KE oblika tetraedra. Linearna interpolacija. CSTh KE

- Matrica raspodele napona KE

$$\mathbf{S} = \mathbf{DB} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_2 & \mathbf{S}_3 & \mathbf{S}_4 \end{bmatrix}$$

$$\mathbf{S}_i = \frac{E}{6V(1+\nu)(1-2\nu)} \begin{bmatrix} b_i - b_i\nu & c_i\nu & d_i\nu \\ b_i\nu & c_i - c_i\nu & d_i\nu \\ b_i\nu & c_i\nu & d_i - d_i\nu \\ \frac{1}{2}c_i(1-2\nu) & \frac{1}{2}b_i(1-2\nu) & 0 \\ 0 & \frac{1}{2}d_i(1-2\nu) & \frac{1}{2}c_i(1-2\nu) \\ \frac{1}{2}d_i(1-2\nu) & 0 & \frac{1}{2}b_i(1-2\nu) \end{bmatrix}, \quad i = 1,2,3,4$$

  - Zaključuje se da su pored deformacije i komponente napona u KE konstantne. Ovo ima za posledicu potrebu za velikim brojem KE na mestima naglih promena deformacije i napona što je mana ovog KE
- Vektor ekvivalentnog opterećenja
  - Ukoliko na KE deluju konstantne zapreminske sile

$$\mathbf{q} = \begin{Bmatrix} q_{x0} = const. \\ q_{y0} = const. \\ q_{z0} = const. \end{Bmatrix}$$

prvi član vektora ekvivalentnog opterećenja

$$\int_V N_1 q_{x0} dV = q_{x0} \int_V N_1^1 N_2^0 N_3^0 N_4^0 dV = q_{x0} \frac{1!0!0!0!}{(1+0+0+0+3)!} 6V = q_{x0} \frac{V}{4}$$

$$\mathbf{Q}^T = \frac{V}{4} \begin{Bmatrix} q_{x0} & q_{y0} & q_{z0} & q_{x0} & q_{y0} & q_{z0} & q_{x0} & q_{y0} & q_{z0} & q_{x0} & q_{y0} & q_{z0} \end{Bmatrix}$$

ukupna zapreminska sila ravnomerno se deli na svaki čvor KE

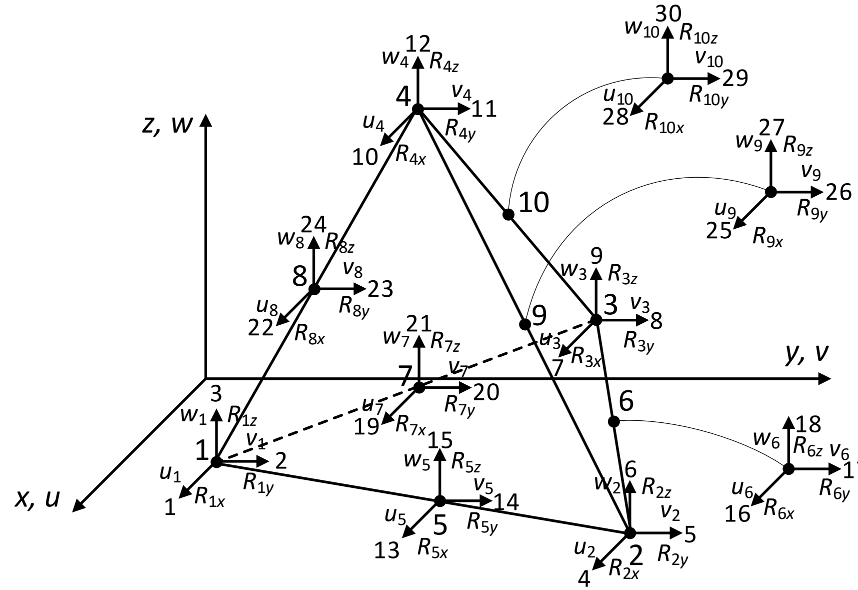
KE oblika tetraedra. Kvadratna interpolacija. LSTh KE

- KE oblika tetraedra sa čvorovima u temenima i sredinama ivica

$$\mathbf{d}^T = \{\mathbf{d}_1 \quad \mathbf{d}_2 \quad \mathbf{d}_3 \quad \mathbf{d}_4 \quad \mathbf{d}_5 \quad \mathbf{d}_6 \quad \mathbf{d}_7 \quad \mathbf{d}_8 \quad \mathbf{d}_9 \quad \mathbf{d}_{10}\}$$

$$\mathbf{d}_i^T = \{u_i \quad v_i \quad w_i\}, \qquad i = 1,2,3,\ldots,10$$

- Raspodela pomeranja u polju KE definisana je potpunim polinomima drugog stepena



$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2 + \alpha_8 xy + \alpha_9 yz + \alpha_{10} xz$$

$$v = \alpha_{11} + \alpha_{12} x + \alpha_{13} y + \alpha_{14} z + \alpha_{15} x^2 + \alpha_{16} y^2 + \alpha_{17} z^2 + \alpha_{18} xy + \alpha_{19} yz + \alpha_{20} xz$$

$$w = \alpha_{21} + \alpha_{22} x + \alpha_{23} y + \alpha_{24} z + \alpha_{25} x^2 + \alpha_{26} y^2 + \alpha_{27} z^2 + \alpha_{28} xy + \alpha_{29} yz + \alpha_{30} xz$$

- Matrica IF

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 & 0 & 0 & N_9 & 0 & 0 & N_{10} & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 & 0 & 0 & N_9 & 0 & 0 & N_{10} & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 & 0 & 0 & N_9 & 0 & 0 & N_{10} \end{bmatrix}$$

KE oblika tetraedra. Kvadratna interpolacija. LSTh KE

- IF. Prirodne zapreminske koordinate

$$L_i = \frac{1}{6V}\left(a_i + b_i x + c_i y + d_i z\right), \qquad i = 1,2,3,4$$

$$N_1 = L_1\left(2L_1 - 1\right), \quad N_2 = L_2\left(2L_2 - 1\right), \quad N_3 = L_3\left(2L_3 - 1\right), \quad N_4 = L_4\left(2L_4 - 1\right)$$

$$N_5 = 4L_1 4L_2, \quad N_6 = 4L_2 4L_3, \quad N_7 = 4L_2 4L_3, \quad N_8 = 4L_1 4L_3$$

$$N_9 = 4L_2 4L_4, \quad N_{10} = 4L_3 4L_4$$

- Matrica **B**

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \cdots & \mathbf{B}_{10} \end{bmatrix} \qquad \mathbf{B}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix}, \qquad i = 1,2,3,\ldots,10$$

KE oblika tetraedra. Kvadratna interpolacija. LSTh KE

- Matrica **B**
  - S obzirom na to da je $N_i = N_i(L_1(x, y, z), L_2(x, y, z), L_3(x, y, z), L_4(x, y, z))$ koristeći lančano pravilo parcijalnog diferenciranja dobija se

$$\left.\begin{aligned}\frac{\partial N_i}{\partial x} &= \frac{\partial N_i}{\partial L_1}\frac{\partial L_1}{\partial x} + \frac{\partial N_i}{\partial L_2}\frac{\partial L_2}{\partial x} + \frac{\partial N_i}{\partial L_3}\frac{\partial L_3}{\partial x} + \frac{\partial N_i}{\partial L_4}\frac{\partial L_4}{\partial x}\\ \frac{\partial N_i}{\partial y} &= \frac{\partial N_i}{\partial L_1}\frac{\partial L_1}{\partial y} + \frac{\partial N_i}{\partial L_2}\frac{\partial L_2}{\partial y} + \frac{\partial N_i}{\partial L_3}\frac{\partial L_3}{\partial y} + \frac{\partial N_i}{\partial L_4}\frac{\partial L_4}{\partial y}\\ \frac{\partial N_i}{\partial z} &= \frac{\partial N_i}{\partial L_1}\frac{\partial L_1}{\partial z} + \frac{\partial N_i}{\partial L_2}\frac{\partial L_2}{\partial z} + \frac{\partial N_i}{\partial L_3}\frac{\partial L_3}{\partial z} + \frac{\partial N_i}{\partial L_4}\frac{\partial L_4}{\partial z}\end{aligned}\right\}, \quad i = 1,2,3,\ldots,10$$

$$\left.\begin{aligned}\frac{\partial L_i}{\partial x} &= \frac{b_i}{6V}\\ \frac{\partial L_i}{\partial y} &= \frac{c_i}{6V}\\ \frac{\partial L_i}{\partial z} &= \frac{d_i}{6V}\end{aligned}\right\}, \quad i = 1,2,3,4$$

$$\left.\begin{aligned}\frac{\partial N_i}{\partial x} &= \frac{1}{6V}\left(\frac{\partial N_i}{\partial L_1}b_1 + \frac{\partial N_i}{\partial L_2}b_2 + \frac{\partial N_i}{\partial L_3}b_3 + \frac{\partial N_i}{\partial L_4}b_4\right)\\ \frac{\partial N_i}{\partial y} &= \frac{1}{6V}\left(\frac{\partial N_i}{\partial L_1}c_1 + \frac{\partial N_i}{\partial L_2}c_2 + \frac{\partial N_i}{\partial L_3}c_3 + \frac{\partial N_i}{\partial L_4}c_4\right)\\ \frac{\partial N_i}{\partial z} &= \frac{1}{6V}\left(\frac{\partial N_i}{\partial L_1}d_1 + \frac{\partial N_i}{\partial L_2}d_2 + \frac{\partial N_i}{\partial L_3}d_3 + \frac{\partial N_i}{\partial L_4}d_4\right)\end{aligned}\right\}, \quad i = 1,2,3,\ldots,10$$

KE oblika tetraedra. Kvadratna interpolacija. LSTh KE

- Nakon određivanja matrice **B** pomoću izraza koristeći matricu elastičnosti i vodeći računa o pravilu za integraciju određuje se matrica krutosti **k**
- U diferencijalnom operatoru pojavljuju se parcijalni izvodi prvog reda pa se zaključuje da je stepen funkcije koja opisuje raspodelu deformacije u polju KE za jedan manji od stepena funkcije koja opisuje raspodelu pomeranja, tj. aproksimacija raspodele deformacije u polju KE je linearna pa se zbog toga element naziva **tetraedar s linearnim deformacijama ili LSTh element** (Linear Strain Tetrahedral element). Razmatrani element **spada u grupu konformnih elemenata**, a **ispunjava i ostale uslove za monotonu konvergenciju rešenja**
- Analognim postupcima opisanim za CSTh element određuje se vektor ekvivalentnog opterećenja, raspodela deformacije i napona za LSTh element

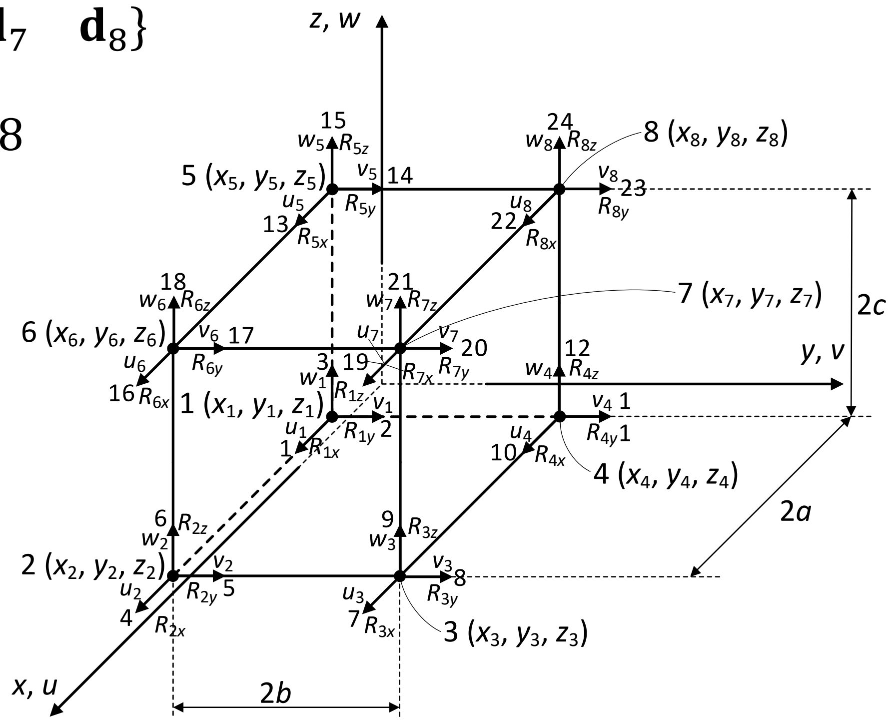
Prizmatični konačni elementi. Pravougaoni paralelopiped

- KE oblika pravougaonog paralelopipeda koji ima 8 čvorova

$$\mathbf{d}^T = \{\mathbf{d}_1 \quad \mathbf{d}_2 \quad \mathbf{d}_3 \quad \mathbf{d}_4 \quad \mathbf{d}_5 \quad \mathbf{d}_6 \quad \mathbf{d}_7 \quad \mathbf{d}_8\}$$

$$\mathbf{d}_i^T = \{u_i \quad v_i \quad w_i\}, \qquad i = 1,2,3,\ldots,8$$

- Raspodela pomeranja u polju KE definisana je nepotpunim polinomima trećeg stepena, tzv. trilinearna interpolacija pomeranja jer IF sadrže proizvode tri linearna polinoma

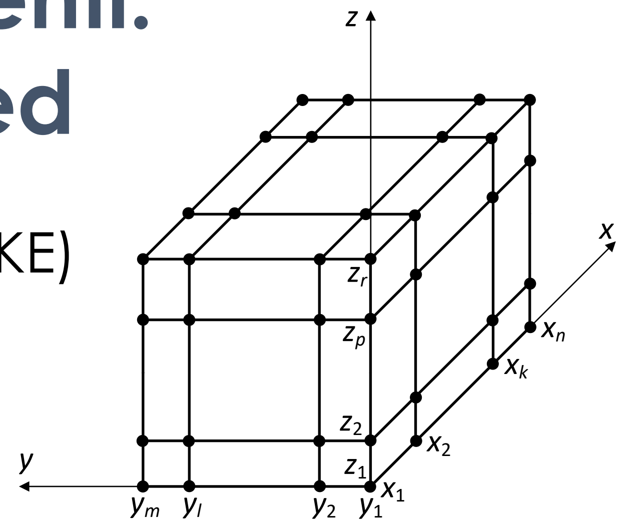


$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 xy + \alpha_6 yz + \alpha_7 zx + \alpha_8 xyz$$

$$v = \alpha_9 + \alpha_{10} x + \alpha_{11} y + \alpha_{12} z + \alpha_{13} xy + \alpha_{14} yz + \alpha_{15} zx + \alpha_{16} xyz$$

$$w = \alpha_{17} + \alpha_{18} x + \alpha_{19} y + \alpha_{20} z + \alpha_{21} xy + \alpha_{22} yz + \alpha_{23} zx + \alpha_{24} xyz$$

Prizmatični konačni elementi. Pravougaoni paralelopiped



- IF (analogno kao i kod pravougaonog KE)
  - Lagranžovi polinomi

$$N_i(x,y) = L_k(x)L_l(y)L_p(z)$$

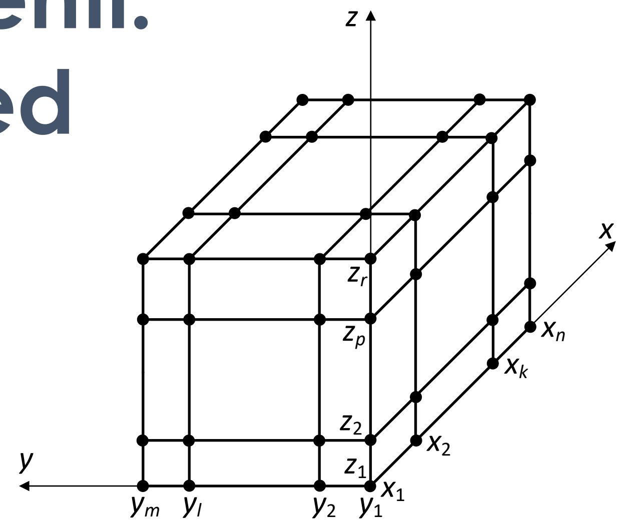
$$L_k(x) = \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{x-x_j}{x_k-x_j}, \quad L_l(y) = \prod_{\substack{j=1 \\ j\neq l}}^{m} \frac{y-y_j}{y_l-y_j}, \quad L_p(z) = \prod_{\substack{j=1 \\ j\neq p}}^{r} \frac{z-z_j}{z_p-z_j}$$

    - gde su n, m i r brojevi čvorova u pravcu osa x, y i z, respektivno i $1 \leq k \leq n$, $1 \leq l \leq m$ i $1 \leq p \leq r$
  - Vodeći računa o koordinatama početnih i krajnjih čvorova duž ivica KE ($x_1 = -a$, $x_2 = a$, $y_1 = -b$, $y_2 = b$, $z_1 = -c$, $z_2 = c$), Lagranžovi polinomi u pravcima osa x, y i z su (slično kao i kod dvodimenzionalnog pravougaonog KE) glase

$$L_1(x) = \frac{1}{2}\left(1-\frac{x}{a}\right), \quad L_1(y) = \frac{1}{2}\left(1-\frac{y}{b}\right), \quad L_1(z) = \frac{1}{2}\left(1-\frac{z}{c}\right)$$

$$L_2(x) = \frac{1}{2}\left(1+\frac{x}{a}\right), \quad L_2(y) = \frac{1}{2}\left(1+\frac{y}{b}\right), \quad L_2(z) = \frac{1}{2}\left(1+\frac{z}{c}\right)$$

Prizmatični konačni elementi. Pravougaoni paralelopiped



- IF

$$N_1 = L_1(x)L_1(y)L_1(z) = \frac{1}{8}\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)\left(1-\frac{z}{c}\right)$$

$$N_2 = L_2(x)L_1(y)L_1(z) = \frac{1}{8}\left(1+\frac{x}{a}\right)\left(1-\frac{y}{b}\right)\left(1-\frac{z}{c}\right)$$

$$N_3 = L_2(x)L_2(y)L_1(z) = \frac{1}{8}\left(1+\frac{x}{a}\right)\left(1+\frac{y}{b}\right)\left(1-\frac{z}{c}\right)$$

$$N_4 = L_1(x)L_2(y)L_1(z) = \frac{1}{8}\left(1-\frac{x}{a}\right)\left(1+\frac{y}{b}\right)\left(1-\frac{z}{c}\right)$$

$$N_5 = L_1(x)L_1(y)L_2(z) = \frac{1}{8}\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)\left(1+\frac{z}{c}\right)$$

$$N_6 = L_2(x)L_1(y)L_2(z) = \frac{1}{8}\left(1+\frac{x}{a}\right)\left(1-\frac{y}{b}\right)\left(1+\frac{z}{c}\right)$$

$$N_7 = L_2(x)L_2(y)L_2(z) = \frac{1}{8}\left(1+\frac{x}{a}\right)\left(1+\frac{y}{b}\right)\left(1+\frac{z}{c}\right)$$

$$N_8 = L_1(x)L_2(y)L_2(z) = \frac{1}{8}\left(1-\frac{x}{a}\right)\left(1+\frac{y}{b}\right)\left(1+\frac{z}{c}\right)$$

$$\mathbf{N} = \left[\begin{array}{cccccccccccc} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{array}\right.$$

$$\left.\begin{array}{cccccccccccc} N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 & 0 & 0 \\ 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 & 0 \\ 0 & 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 \end{array}\right]$$

- Matrica krutosti

$$\mathbf{k} = \int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a} \mathbf{B}^T\mathbf{D}\mathbf{B}\,dxdydz$$

Prizmatični konačni elementi. Pravougaoni paralelopiped

- IF. Prirodne koordinate

$$N_i = \frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta), \qquad i = 1,2,3,\ldots,8$$

  - gde su $\xi_i = \pm 1$, $\eta_i = \pm 1$ i $\zeta_i = \pm 1$ koordinate čvorova, a veze između prirodnih i lokalnih Dekartovih koordinata glase

$$\xi = \frac{x}{a}, \qquad \eta = \frac{y}{b}, \qquad \zeta = \frac{z}{c}$$

- Sada matrica krutosti glasi

$$\mathbf{k} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T\mathbf{D}\mathbf{B}\det\mathbf{J}\,d\xi d\eta d\zeta$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

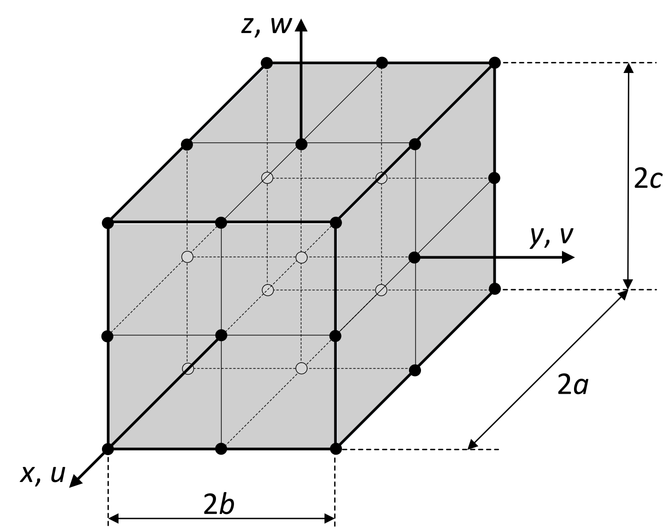
$$\det\mathbf{J} = abc$$

Prizmatični konačni elementi. Pravougaoni paralelopiped

- Pri određivanju matrice **B** neophodno je odrediti parcijalne izvode IF po lokalnim Dekartovim koordinatama

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial z} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{a} & 0 & 0 \\ 0 & \dfrac{1}{b} & 0 \\ 0 & 0 & \dfrac{1}{c} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \\ \dfrac{\partial N_i}{\partial \zeta} \end{Bmatrix}, \qquad i = 1,2,3,\ldots,8$$

- **Lagranžov KE drugog reda (27 čvorova i 81 stepen slobode)**



Prizmatični konačni elementi. Pravougaoni paralelopiped

- **Lagranžov KE drugog reda**
  - Lagranžovi polinomi za čvorove u pravcu osa x, y i z

$$L_1(x)=\frac{x-x_2}{x_1-x_2}\frac{x-x_3}{x_1-x_3},\quad L_2(x)=\frac{x-x_1}{x_2-x_1}\frac{x-x_3}{x_2-x_3},\quad L_3(x)=\frac{x-x_1}{x_3-x_1}\frac{x-x_2}{x_3-x_2}$$

$$L_1(y)=\frac{y-y_2}{y_1-y_2}\frac{y-y_3}{y_1-y_3},\quad L_2(y)=\frac{y-y_1}{y_2-y_1}\frac{y-y_3}{y_2-y_3},\quad L_3(y)=\frac{y-y_1}{y_3-y_1}\frac{y-y_2}{y_3-y_2}$$

$$L_1(z)=\frac{z-z_2}{z_1-z_2}\frac{z-z_3}{z_1-z_3},\quad L_2(z)=\frac{z-z_1}{z_2-z_1}\frac{y-z_3}{z_2-z_3},\quad L_3(z)=\frac{z-z_1}{z_3-z_1}\frac{z-z_2}{z_3-z_2}$$

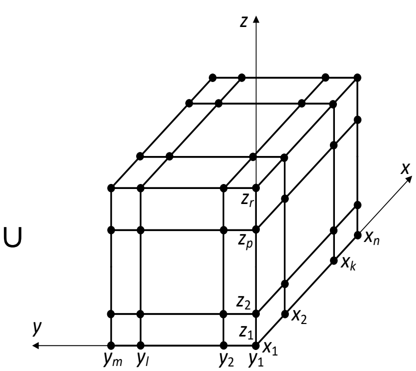
  - odnosno vodeći računa o koordinatama čvorova $x_1 = -a$, $x_2 = 0$, $x_3 = a$, $y_1 = -b$, $y_2 = 0$, $y_3 = b$, $z_1 = -c$, $z_2 = 0$ i $z_3 = b$ sledi

$$L_1(x)=\frac{1}{2}\left(\frac{x^2}{a^2}-\frac{x}{a}\right),\quad L_2(x)=1-\frac{x^2}{a^2},\quad L_3(x)=\frac{1}{2}\left(\frac{x^2}{a^2}+\frac{x}{a}\right)$$

$$L_1(y)=\frac{1}{2}\left(\frac{y^2}{b^2}-\frac{y}{b}\right),\quad L_2(y)=1-\frac{y^2}{b^2},\quad L_3(y)=\frac{1}{2}\left(\frac{y^2}{b^2}+\frac{y}{b}\right)$$

$$L_1(z)=\frac{1}{2}\left(\frac{z^2}{c^2}-\frac{z}{c}\right),\quad L_2(z)=1-\frac{z^2}{c^2},\quad L_3(z)=\frac{1}{2}\left(\frac{z^2}{c^2}+\frac{z}{c}\right)$$

Analognim postupkom kao i kod pravougaonog KE koristeći šemu i izraz $N_i(x,y,z) = L_k(x)L_l(y)L_p(z)$ određuju se IF



Prizmatični konačni elementi. Pravougaoni paralelopiped

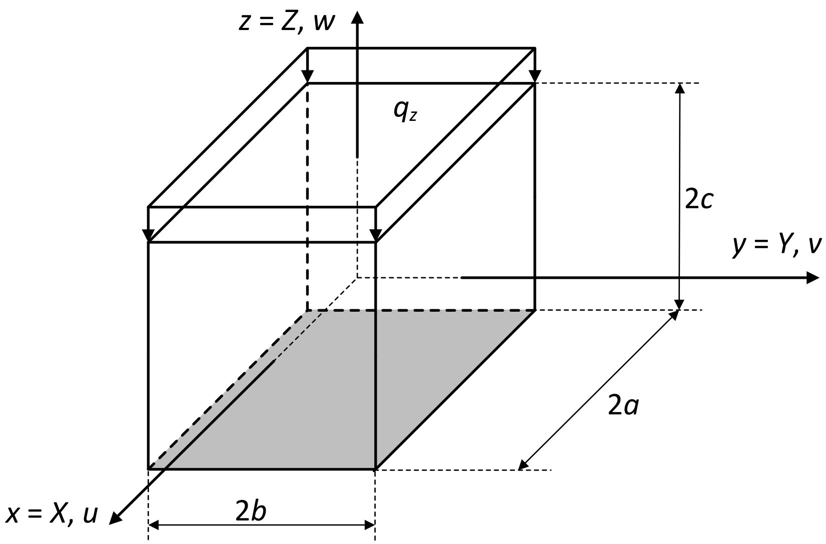
- S obzirom na to da Lagranžovi KE višeg reda imaju unutrašnje čvorove, kod njih se javlja veliki broj stepeni slobode
- Slično kao i kod dvodimenzionalnih KE, nepotpunost polinoma za aproksimaciju pomeranja u polju trodimenzionalnog Lagranžovog KE je nedostatak ali im je dobra osobina jednostavnost izvođenja IF
- Pored elemenata sa jednakim brojem čvorova u pravcu Dekartovih osa mogu da se izvedu elementi s različitim brojem čvorova u pravcu koordinatnih osa množenjem Lagranžovih polinoma različitog stepena

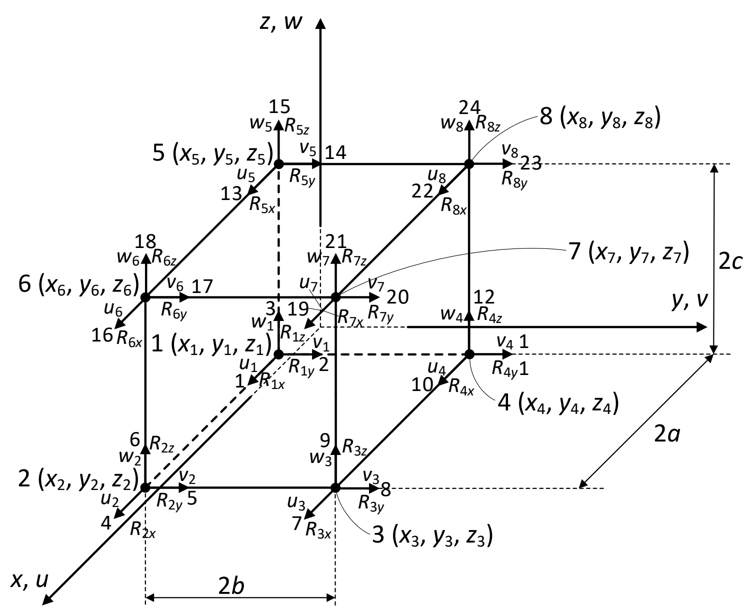
Prizmatični konačni elementi. Pravougaoni paralelopiped

- **Primer**
  - Podaci su: q_z = 10 MPa, modul elastičnosti E = 30 GPa, Poasonov koeficijent ν = 0,2, a = b = c = 0,5 m

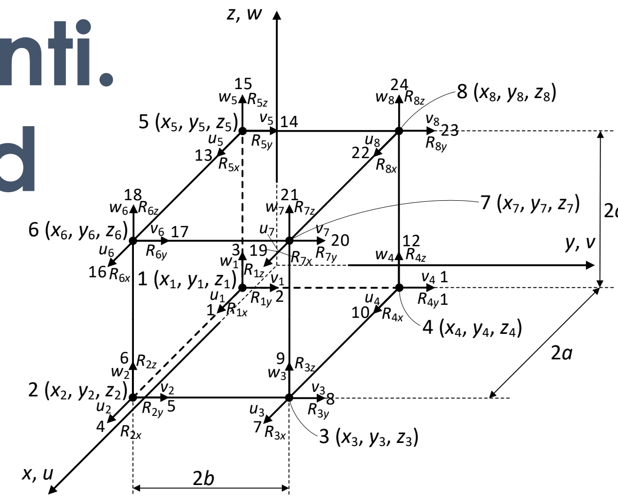
Matematički model 3D tela oslonjenog u svim pravcima po površini jedne stranice (zatamnjena površina). Potrebno je odrediti pomeranje temena opterećene stranice

Matematički model je diskretizovan jednim 3D KE koji ima 8 čvorova. Postoji simetrija ali nije korišćena pri analizi





Prizmatični konačni elementi. Pravougaoni paralelopiped



- **Primer**

$$\mathbf{d}=\mathbf{d}^{*}=\begin{Bmatrix} d_1^{*}\\ \vdots \\ d_8^{*}\end{Bmatrix}=\left\{\begin{array}{c|c} u_1 & 1\\ v_1 & 2\\ w_1 & 3\\ \vdots & \vdots\\ u_8 & 22\\ v_8 & 23\\ w_8 & 24\end{array}\right\} \qquad \mathbf{d}_a^{*}=\left\{\begin{array}{c|c} u_5 & 13\\ v_5 & 14\\ w_5 & 15\\ \vdots & \vdots\\ u_8 & 22\\ v_8 & 23\\ w_8 & 24\end{array}\right\}$$

$$\mathbf{K}_{aa}^{*}\mathbf{d}_a^{*}=\mathbf{S}_a^{*}=\mathbf{P}_a^{*}+\mathbf{Q}_a^{*}\Rightarrow \mathbf{d}_a^{*}=\mathbf{K}_{aa}^{*-1}\mathbf{S}_a^{*}$$

$$\mathbf{D}=\begin{bmatrix} 3,\dot{3} & 0,8\dot{3} & 0,8\dot{3} & 0 & 0 & 0\\ & 3,\dot{3} & 0,8\dot{3} & 0 & 0 & 0\\ & & 3,\dot{3} & 0 & 0 & 0\\ & & & 1,25 & 0 & 0\\ & & & & 1,25 & 0\\ \text{sim.} & & & & & 1,25\end{bmatrix}\cdot 10^{7}$$

Matrica **B** je prikazana samo za jedna čvor

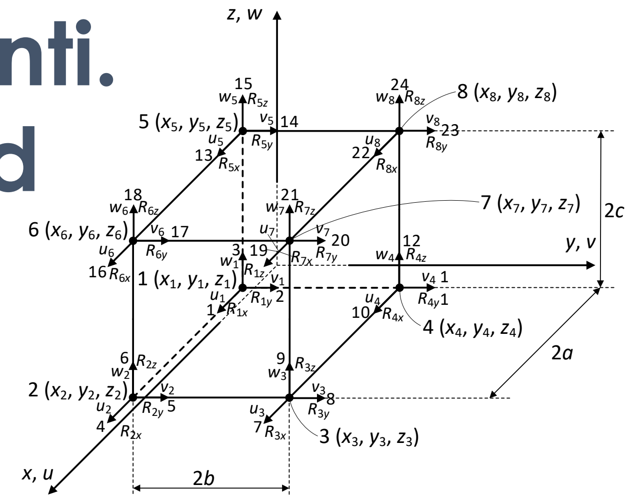
$$\mathbf{B}_1=\begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & 0\\ 0 & \frac{\partial N_1}{\partial y} & 0\\ 0 & 0 & \frac{\partial N_1}{\partial z}\\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0\\ 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y}\\ \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x}\end{bmatrix}$$

$$\frac{\partial N_1}{\partial x}=-\frac{(b-y)(c-z)}{8abc}=-0,25(1-2y)(1-2z)$$

$$\frac{\partial N_1}{\partial y}=-\frac{(a-x)(c-z)}{8abc}=-0,25(1-2x)(1-2z)$$

$$\frac{\partial N_1}{\partial z}=-\frac{(a-x)(b-y)}{8abc}=-0,25(1-2x)(1-2y)$$

Prizmatični konačni elementi. Pravougaoni paralelopiped



- **Primer**

$$
\mathbf{k}=\mathbf{k}^{*}=
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & \cdots & 23 & 24 \end{array} \\
\left[\begin{array}{ccccc}
6{,}4815 & 1{,}7361 & \cdots & 0{,}1736 & 0{,}1736 \\
 & 6{,}4815 & \cdots & -2{,}1991 & -1{,}7361 \\
 & & \ddots & \vdots & \vdots \\
 & & & 6{,}4815 & 1{,}7361 \\
\text{sim.} & & & & 6{,}4815
\end{array}\right]_{24\times 24}
\end{array}
\begin{array}{c} 1 \\ 2 \\ \vdots \\ 23 \\ 24 \end{array}
\cdot 10^{6}
$$

IF za opterećenu površinu (z = c = 0,5 m) glase

$$N_{s1}=N_{s2}=N_{s3}=N_{s4}=0$$

$$N_{s5}=\frac{0{,}25(a-x)(b-y)}{ab}=0{,}25(1-2x)(1-2y)$$

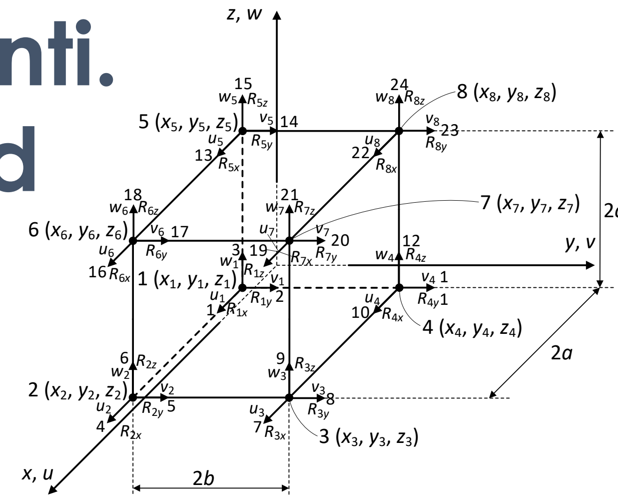
$$N_{s6}=\frac{0{,}25(a+x)(b-y)}{ab}=0{,}25(1+2x)(1-2y)$$

$$N_{s7}=\frac{0{,}25(a+x)(b+y)}{ab}=0{,}25(1+2x)(1+2y)$$

$$N_{s8}=\frac{0{,}25(a-x)(b+y)}{ab}=0{,}25(1-2x)(1+2y)$$

$$
\begin{Bmatrix} Q_{5z} \\ Q_{6z} \\ Q_{7z} \\ Q_{8z} \end{Bmatrix}
=
\begin{Bmatrix} Q_{5z}^{*} \\ Q_{6z}^{*} \\ Q_{7z}^{*} \\ Q_{8z}^{*} \end{Bmatrix}
=\int_{-b}^{-b}\int_{-a}^{-a}
\begin{Bmatrix} N_{s5}(x,y) \\ N_{s6}(x,y) \\ N_{s7}(x,y) \\ N_{s8}(x,y) \end{Bmatrix}
q_z(x,y)\,dxdy=-
\begin{Bmatrix} 2500{,}0 \\ 2500{,}0 \\ 2500{,}0 \\ 2500{,}0 \end{Bmatrix}
\begin{array}{c} 15 \\ 18 \\ 21 \\ 24 \end{array}
$$

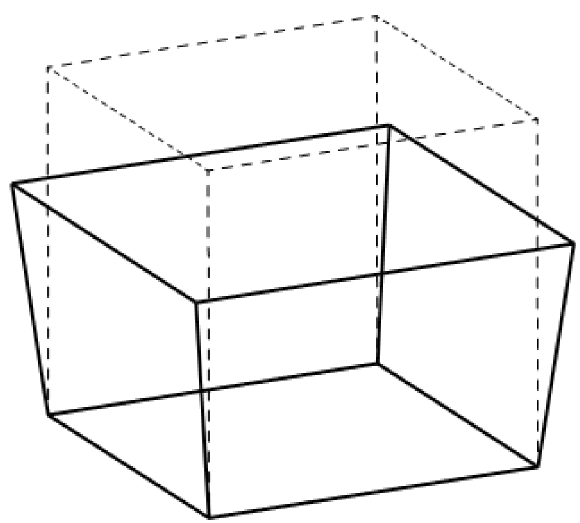
Prizmatični konačni elementi. Pravougaoni paralelopiped



Primer

$$
\mathbf{K}_{aa}^{*} = \begin{array}{c}
\begin{array}{ccccc} 13 & 14 & \cdots & 23 & 24 \end{array} \\
\left[\begin{array}{ccccc|c}
6{,}4815 & 1{,}7361 & \cdots & 0{,}3472 & -0{,}8681 & 13 \\
 & 6{,}4815 & \cdots & -2{,}3148 & 0{,}3472 & 14 \\
 & & \ddots & \vdots & \vdots & \vdots \\
 & & & 6{,}4815 & 1{,}7361 & 23 \\
\text{sim.} & & & & 6{,}4815 & 24
\end{array}\right]_{12\text{x}12}
\end{array} \cdot 10^{6}
\qquad
\mathbf{Q}_{a}^{*} = \mathbf{S}_{a}^{*} = -\left\{\begin{array}{c|c}
0 & 13 \\
0 & 14 \\
2500{,}0 & 15 \\
\vdots & \vdots \\
0 & 22 \\
0 & 23 \\
2500{,}0 & 24
\end{array}\right\}_{12\text{x}1}
$$

$$
\mathbf{d}_{a}^{*} = \mathbf{K}_{aa}^{*-1}\mathbf{S}_{a}^{*} = \left\{\begin{array}{c|c}
-0{,}0450 & 13 \\
-0{,}0450 & 14 \\
-0{,}3225 & 15 \\
0{,}0450 & 16 \\
-0{,}0450 & 17 \\
-0{,}3225 & 18 \\
0{,}0450 & 19 \\
0{,}0450 & 20 \\
-0{,}3225 & 21 \\
-0{,}0450 & 22 \\
0{,}0450 & 23 \\
-0{,}3225 & 24
\end{array}\right\} \cdot 10^{-3}\,\text{m}
$$



Komentari:

- Konvergencija ka tačnom rešenju može da se postigne povećanjem broja KE u modelu (progušćenje mreže) i/ili primenom KE višeg reda
- Pomeranje temena opterećene stranice u pravcu z ose, određeno primenom KE oblika pravougaonog paralelopipeda koji imaju 8, 20, 21 i 27 čvorova i KE oblika tetraedra koji imaju 4, 10 i 11 čvorova u modelima sa uniformnom i adaptivnom mrežom, konvergira ka vrednosti približno jednakoj – 0,331 · 10^{-3} m

Prizmatični konačni elementi. Pravougaoni paralelopiped

- **Primer**
  - Primena prirodnih koordinata pojednostavljuje određivanje matrice krutosti KE i vektora ekvivalentnog opterećenja

$$N_i = \frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta), \qquad i = 1,2,3,\ldots,8$$

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial z} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{a} & 0 & 0 \\ 0 & \dfrac{1}{b} & 0 \\ 0 & 0 & \dfrac{1}{c} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \\ \dfrac{\partial N_i}{\partial \zeta} \end{Bmatrix}, \qquad i = 1,2,3,\ldots,8$$

$$\mathbf{k} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T \mathbf{D} \mathbf{B} \det \mathbf{J} \, d\xi \, d\eta \, d\zeta \qquad \det \mathbf{J} = abc$$

Vodeći računa da je za opterećenu površinu prirodna koordinata $\zeta = 1$

$$\begin{Bmatrix} Q_{5z} \\ Q_{6z} \\ Q_{7z} \\ Q_{8z} \end{Bmatrix} = \int_{-1}^{1}\int_{-1}^{1} \begin{Bmatrix} N_{s5}(\xi,\eta) \\ N_{s6}(\xi,\eta) \\ N_{s7}(\xi,\eta) \\ N_{s8}(\xi,\eta) \end{Bmatrix} q_z(\xi,\eta) \det \mathbf{J} \, d\xi \, d\eta \qquad \det \mathbf{J} = ab$$

Prizmatični konačni elementi. *Serendipiti* KE

- Slično kao i kod dvodimenzionalnih KE, imaju čvorove samo duž ivica i na taj način smanjuje se broj stepeni slobode u odnosu na Lagranžove KE pri istoj raspodeli pomeranja
- Prednost u odnosu na Lagranžove KE je i u manjoj nepotpunosti polinoma ali imaju složeniji postupak izvođenja IF
- U Serendipiti familiju spada i KE u obliku pravougaonog paralelopipeda, tzv. linearni element (8 čvorova i 24 stepena slobode)

Prizmatični konačni elementi. *Serendipiti* KE

- **Kvadratni (paraboličan) element (20 čvorova i 60 stepeni slobode)**
- IF
  - za čvorove u temenima

$$N_i = \frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta)(\xi_i\xi+\eta_i\eta+\zeta_i\zeta-2)$$

    - gde su $\xi_i = \pm 1$, $\eta_i = \pm 1$ i $\zeta_i = \pm 1$ koordinate čvorova
  - i za čvorove u sredinama ivica

$\xi_i = 0$, $\eta_i = \pm 1$ i $\zeta_i = \pm 1$:

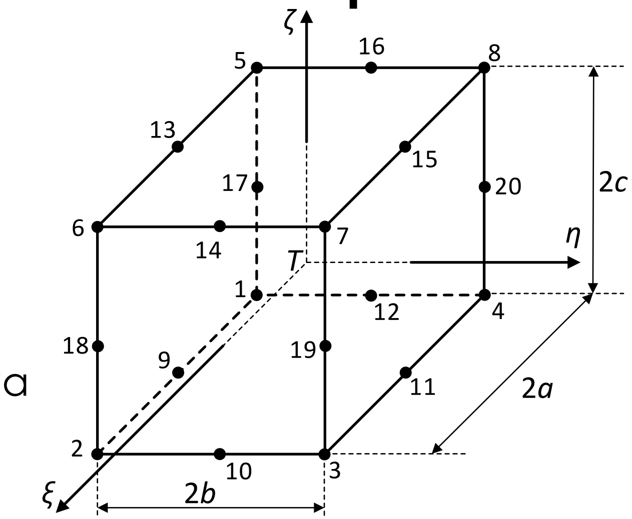
$$N_i = \frac{1}{4}(1-\xi^2)(1+\eta_i\eta)(1+\zeta_i\zeta)$$

$\xi_i = \pm 1$, $\eta_i = 0$ i $\zeta_i = \pm 1$:

$$N_i = \frac{1}{4}(1+\xi_i\xi)(1-\eta^2)(1+\zeta_i\zeta)$$

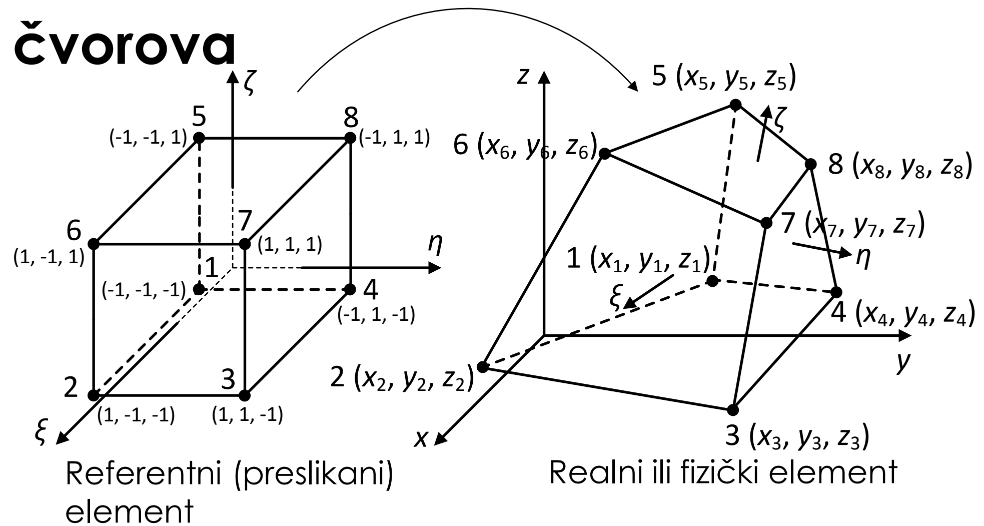
i $\xi_i = \pm 1$, $\eta_i = \pm 1$ i $\zeta_i = 0$:

$$N_i = \frac{1}{4}(1+\xi_i\xi)(1+\eta^2)(1-\zeta^2)$$



Izoparametarski konačni elementi. KE oblika heksaedra

- **Heksaedarski KE koji ima 8 čvorova**
- Geometrija i raspodela pomeranja proizvoljnog heksaedarskog KE definišu se na sledeći način



$$x=\sum_{i=1}^{8}N_i(\xi,\eta,\zeta)x_i,\quad y=\sum_{i=1}^{8}N_i(\xi,\eta,\zeta)y_i,\quad z=\sum_{i=1}^{8}N_i(\xi,\eta,\zeta)z_i$$

$$u=\sum_{i=1}^{8}N_i(\xi,\eta,\zeta)u_i,\quad v=\sum_{i=1}^{8}N_i(\xi,\eta,\zeta)v_i,\quad w=\sum_{i=1}^{8}N_i(\xi,\eta,\zeta)w_i$$

  - gde su x_i, y_i i z_i Dekartove koordinate *i*-tog čvora realnog elementa, a u_i, v_i i w_i komponente pomeranja *i*-tog čvora realnog elementa u pravcu Dekartovih osa
- Funkcije preslikavanja i IF su

$$N_i=\frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta),\qquad i=1,2,3,\ldots,8$$

Izoparametarski konačni elementi. KE oblika heksaedra

- Parcijalni izvodi IF po Dekartovim koordinatama određuju se prema izrazu

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{Bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N_{čvor}} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{N_{čvor}} \frac{\partial N_i}{\partial \xi} y_i & \sum_{i=1}^{N_{čvor}} \frac{\partial N_i}{\partial \xi} z_i \\ \sum_{i=1}^{N_{čvor}} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{N_{čvor}} \frac{\partial N_i}{\partial \eta} y_i & \sum_{i=1}^{N_{čvor}} \frac{\partial N_i}{\partial \eta} z_i \\ \sum_{i=1}^{N_{čvor}} \frac{\partial N_i}{\partial \zeta} x_i & \sum_{i=1}^{N_{čvor}} \frac{\partial N_i}{\partial \zeta} y_i & \sum_{i=1}^{N_{čvor}} \frac{\partial N_i}{\partial \zeta} z_i \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22}J_{33} - J_{23}J_{32} & J_{13}J_{32} - J_{12}J_{33} & J_{12}J_{23} - J_{13}J_{22} \\ J_{23}J_{31} - J_{21}J_{33} & J_{11}J_{33} - J_{13}J_{31} & J_{13}J_{21} - J_{11}J_{23} \\ J_{21}J_{32} - J_{22}J_{31} & J_{12}J_{31} - J_{11}J_{32} & J_{11}J_{22} - J_{12}J_{21} \end{bmatrix}$$

$$\det \mathbf{J} = -J_{13}J_{22}J_{31} + J_{12}J_{23}J_{31} + J_{13}J_{21}J_{32} - J_{11}J_{23}J_{32} - J_{12}J_{21}J_{33} + J_{11}J_{22}J_{33}$$

$$dxdydz = \det \mathbf{J} d\xi d\eta d\zeta$$

- **Matrica krutosti**

$$\mathbf{k} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T \mathbf{D} \mathbf{B} \det \mathbf{J} d\xi d\eta d\zeta$$

Zbog složenosti podintegralnih funkcija elementi matrice krutosti određuju se najčešće numeričkom integracijom

Izoparametarski konačni elementi. KE oblika heksaedra

- **Matrica B**

$$\mathbf{B}=\begin{bmatrix}\mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \cdots & \mathbf{B}_{N_{čvor}}\end{bmatrix}$$

$$\mathbf{B}_i=\begin{bmatrix}\frac{\partial N_i}{\partial x} & 0 & 0\\ 0 & \frac{\partial N_i}{\partial y} & 0\\ 0 & 0 & \frac{\partial N_i}{\partial z}\\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0\\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y}\\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x}\end{bmatrix},\qquad i=1,2,3,\ldots,N_{čvor}$$

- **IF za heksaedarski KE koji ima 8 do 20 čvorova**

$$N_1=g_1-\frac{g_9+g_{12}+g_{17}}{2},\quad N_2=g_2-\frac{g_9+g_{10}+g_{18}}{2},\quad N_3=g_3-\frac{g_{10}+g_{11}+g_{19}}{2}$$

$$N_4=g_4-\frac{g_{11}+g_{12}+g_{20}}{2},\quad N_5=g_5-\frac{g_{13}+g_{16}+g_{17}}{2},\quad N_6=g_6-\frac{g_{13}+g_{14}+g_{18}}{2}$$

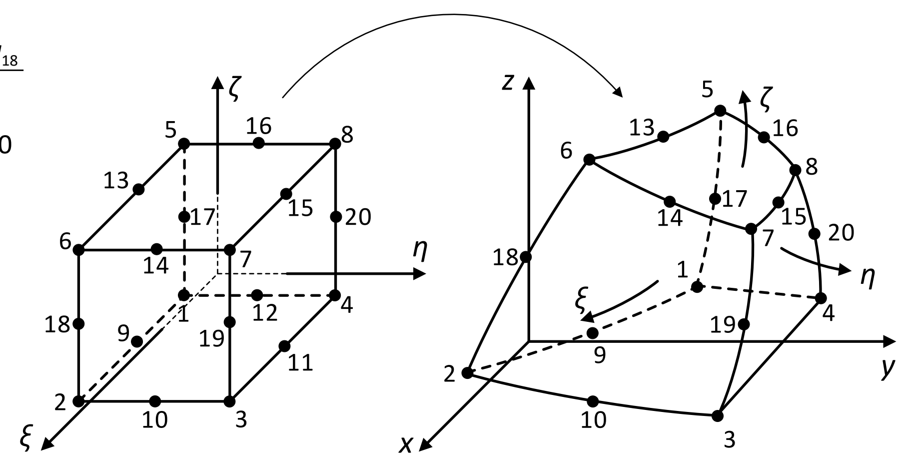
$$N_7=g_7-\frac{g_{14}+g_{15}+g_{19}}{2},\quad N_8=g_8-\frac{g_{15}+g_{16}+g_{20}}{2}\quad N_j=g_j,\ j=9,10,\ldots,20$$

$$g_i=N(\xi,\xi_i)N(\eta,\eta_i)N(\zeta,\zeta_i)$$

$$\xi=\pm1\rightarrow N=\frac{1}{2}(1+\xi\xi_i),\qquad \xi=0\rightarrow N=\frac{1}{2}(1-\xi^2)$$

$$\eta=\pm1\rightarrow N=\frac{1}{2}(1+\eta\eta_i),\qquad \eta=0\rightarrow N=\frac{1}{2}(1-\eta^2)$$

$$\zeta=\pm1\rightarrow N=\frac{1}{2}(1+\zeta\zeta_i),\qquad \zeta=0\rightarrow N=\frac{1}{2}(1-\zeta^2)$$

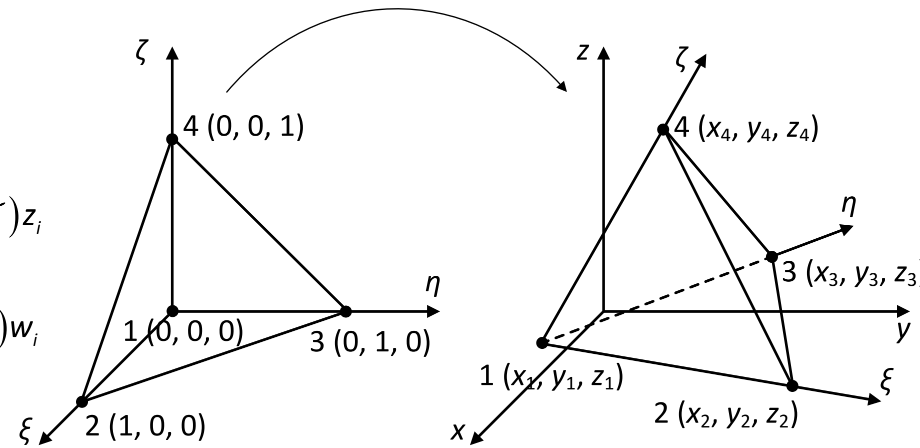


Izoparametarski konačni elementi. KE oblika tetraedra sa 4 čvora

- **Geometrija i raspodela pomeranja opisani su sledećim izrazima**

$$x=\sum_{i=1}^{4}N_i(\xi,\eta,\zeta)x_i,\quad y=\sum_{i=1}^{4}N_i(\xi,\eta,\zeta)y_i,\quad z=\sum_{i=1}^{4}N_i(\xi,\eta,\zeta)z_i$$

$$u=\sum_{i=1}^{4}N_i(\xi,\eta,\zeta)u_i,\quad v=\sum_{i=1}^{4}N_i(\xi,\eta,\zeta)v_i,\quad w=\sum_{i=1}^{4}N_i(\xi,\eta,\zeta)w_i$$



Referentni (preslikani) element **Realni ili fizički element**

- **IF su jednake zapreminskim koordinatama**

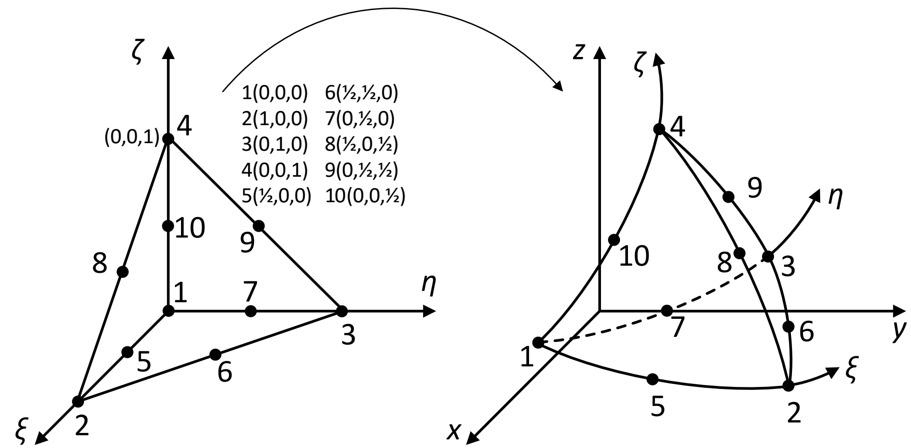
$$N_1=L_1=1-\xi-\eta-\zeta,\qquad N_2=L_2=\xi$$
$$N_3=L_3=\eta,\qquad N_4=L_4=\zeta$$

Izoparametarski konačni elementi. KE oblika tetraedra sa 10 čvorova

- **ili KE drugog reda oblika tetraedra**
- **Geometrija i raspodela pomeranja opisani su sledećim izrazima**

$$x=\sum_{i=1}^{10}N_i(\xi,\eta,\zeta)x_i,\quad y=\sum_{i=1}^{10}N_i(\xi,\eta,\zeta)y_i,\quad z=\sum_{i=1}^{10}N_i(\xi,\eta,\zeta)z_i$$

$$u=\sum_{i=1}^{10}N_i(\xi,\eta,\zeta)u_i,\quad v=\sum_{i=1}^{10}N_i(\xi,\eta,\zeta)v_i,\quad w=\sum_{i=1}^{10}N_i(\xi,\eta,\zeta)w_i$$



Referentni (preslikani) element

Realni ili fizički element

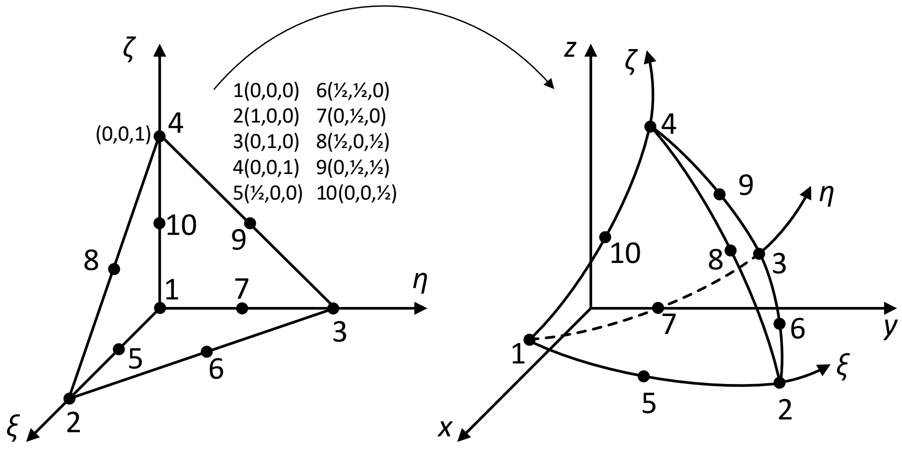
- **IF**

$$N_1=L_1(2L_1-1),\qquad N_2=L_2(2L_2-1)$$
$$N_3=L_3(2L_3-1),\qquad N_4=L_4(2L_4-1)$$
$$N_5=4L_1L_2,\qquad N_6=4L_2\,4L_3,\qquad N_7=4L_2\,4L_3$$
$$N_8=4L_2\,4L_4,\qquad N_9=4L_3\,4L_4,\qquad N_{10}=4L_1\,4L_4$$

$$L_1=1-\xi-\eta-\zeta,\qquad L_2=\xi$$
$$L_3=\eta,\qquad L_4=\zeta$$

Izoparametarski konačni elementi. KE oblika tetraedra

- **IF za KE od 4 do 10 čvorova**



$$N_1 = 1-\xi-\eta-\zeta-\frac{1}{2}N_5-\frac{1}{2}N_7-\frac{1}{2}N_{10}$$

$$N_2 = \xi-\frac{1}{2}N_5-\frac{1}{2}N_6-\frac{1}{2}N_8$$

$$N_3 = \eta-\frac{1}{2}N_6-\frac{1}{2}N_7-\frac{1}{2}N_9$$

$$N_4 = \zeta-\frac{1}{2}N_8-\frac{1}{2}N_9-\frac{1}{2}N_{10}$$

$$N_5 = 4\xi\left(1-\xi-\eta-\zeta\right), \quad N_6 = 4\xi\eta, \quad N_7 = 4\xi\left(1-\xi-\eta-\zeta\right)$$

$$N_8 = 4\xi\zeta, \quad N_9 = 4\eta\zeta, \quad N_{10} = 4\zeta\left(1-\xi-\eta-\zeta\right)$$

- Parcijalni izvodi

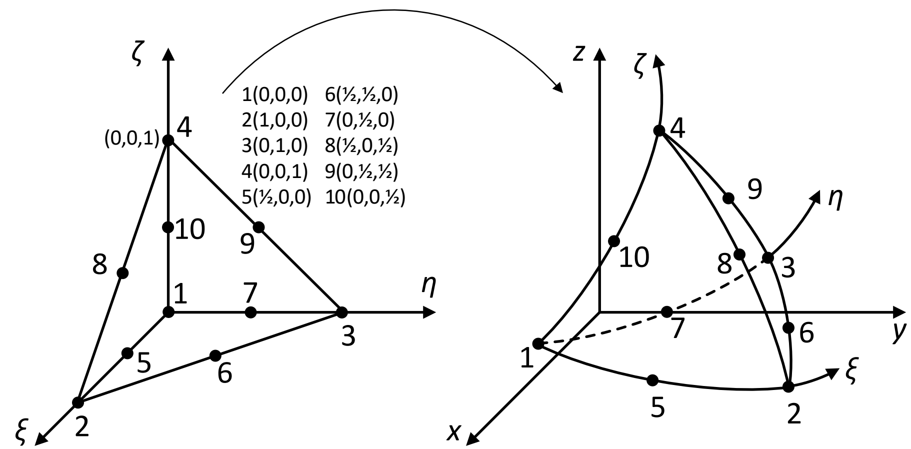
$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \\ \dfrac{\partial N_i}{\partial \zeta} \end{Bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\ \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial y}{\partial \zeta} & \dfrac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N_{čvor}} \dfrac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{N_{čvor}} \dfrac{\partial N_i}{\partial \xi} y_i & \sum_{i=1}^{N_{čvor}} \dfrac{\partial N_i}{\partial \xi} z_i \\ \sum_{i=1}^{N_{čvor}} \dfrac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{N_{čvor}} \dfrac{\partial N_i}{\partial \eta} y_i & \sum_{i=1}^{N_{čvor}} \dfrac{\partial N_i}{\partial \eta} z_i \\ \sum_{i=1}^{N_{čvor}} \dfrac{\partial N_i}{\partial \zeta} x_i & \sum_{i=1}^{N_{čvor}} \dfrac{\partial N_i}{\partial \zeta} y_i & \sum_{i=1}^{N_{čvor}} \dfrac{\partial N_i}{\partial \zeta} z_i \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

$$\mathbf{J}^{-1} = \frac{1}{\det\mathbf{J}} \begin{bmatrix} J_{22}J_{33}-J_{23}J_{32} & J_{13}J_{32}-J_{12}J_{33} & J_{12}J_{23}-J_{13}J_{22} \\ J_{23}J_{31}-J_{21}J_{33} & J_{11}J_{33}-J_{13}J_{31} & J_{13}J_{21}-J_{11}J_{23} \\ J_{21}J_{32}-J_{22}J_{31} & J_{12}J_{31}-J_{11}J_{32} & J_{11}J_{22}-J_{12}J_{21} \end{bmatrix}$$

$$\det\mathbf{J} = -J_{13}J_{22}J_{31}+J_{12}J_{23}J_{31}+J_{13}J_{21}J_{32}-J_{11}J_{23}J_{32}-J_{12}J_{21}J_{33}+J_{11}J_{22}J_{33}$$

Izoparametarski konačni elementi. KE oblika tetraedra



- **Matrica B**

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \cdots & \mathbf{B}_{N_{čvor}} \end{bmatrix}$$

$$\mathbf{B}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix}, \qquad i = 1,2,3,\ldots,N_{čvor}$$

- **Matrica krutosti**

$$\mathbf{k} = \int_{\xi=0}^{\xi=1} \int_{\eta=0}^{\eta=1-\xi} \int_{\zeta=0}^{\zeta=1-\xi-\eta} \mathbf{B}^T \mathbf{D} \mathbf{B} \det \mathbf{J} \, d\zeta \, d\eta \, d\xi$$

Zbog složenosti podintegralnih funkcija elementi matrice krutosti određuju se najčešće **numeričkom integracijom**